



UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

***K*-teoría Hermitiana Algebraica Bivariante**

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires
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Santiago Javier Vega

Director de tesis y Consejero de estudios: Dr. Guillermo Cortiñas

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Hermitian Bivariant Algebraic K -theory
Summary

Consider a commutative ring ℓ with involution with an element $\lambda \in \ell$ such that $\lambda + \lambda^* = 1$; write Alg_ℓ^* for the category of ℓ -algebras with involution compatible with that of ℓ , which we call $*$ -algebras. In this thesis we develop a triangulated category kk^h and a functor $j^h : Alg_\ell^* \rightarrow kk^h$ which we call *bivariant algebraic hermitian K -theory*; the functor j^h satisfies homotopy invariance, matrix and hermitian stability and is an excisive homology theory for extensions which are linearly split.

We also define a Weibel style homotopy invariant hermitian K -theory which we denote as KH_*^h . We show that the category kk^h recovers KH_0^h as a representable functor

$$\mathrm{hom}_{kk^h}(\ell, A) \cong KH_0^h(A).$$

We construct functors ${}_\varepsilon U$ and ${}_\varepsilon V$ which correspond to desuspensions of the functors U' and V' in Karoubi's Fundamental Theorem: for a unital $R \in Alg_\ell^*$ there is an element $\theta_0 \in K_2^h(U'^2 R)$ which the cup product induces an isomorphism

$${}_\varepsilon K_*^h(V'(R)) \cong -{}_\varepsilon K_{*+1}^h(U'(R)).$$

We prove an adjunction between kk^h and the bivariant algebraic K -theory kk as defined by Cortiñas and Thom and use it to prove a version of Karoubi's theorem in kk^h : the product with the image of θ_0 in $KH_0^h(U^2 \ell)$ induces an isomorphism in kk^h

$$j^h({}_\varepsilon V A) \cong \Omega j^h(-{}_\varepsilon U A)$$

for any $A \in Alg_\ell^*$. This allows us to obtain a bivariant homotopic version of the classical 12-term exact sequence of Karoubi for hermitian K -theory.

Keywords: hermitian algebraic K -theory, Karoubi's fundamental theorem, homotopy hermitian K -theory, bivariant algebraic K -theory, bivariant Witt groups

K-teoría Algebraica Hermitiana Bivariante
Resumen

Consideremos un anillo conmutativo ℓ con involución con un elemento $\lambda \in \ell$ tal que $\lambda + \lambda^* = 1$; sea Alg_ℓ^* la categoría de ℓ -álgebras con involución compatible con la de ℓ que llamamos $*$ -álgebras. En esta tesis desarrollamos una categoría triangulada kk^h y un funtor $j^h : Alg_\ell^* \rightarrow kk^h$ que llamamos *K-teoría hermitiana algebraica bivariante*; el funtor j^h satisface invarianza homotópica, estabilidad matricial y hermitiana y es una teoría de homología escisiva para extensiones que se parten linealmente.

También definimos una versión invariante homotópica estilo Weibel de la *K*-teoría hermitiana que notamos como KH_*^h . Mostramos que la categoría kk^h recupera KH_0^h como funtor representable

$$\text{hom}_{kk^h}(\ell, A) \cong KH_0^h(A).$$

Construimos funtores ${}_\varepsilon U$ y ${}_\varepsilon V$ que se corresponden con desuspensiones de los funtores U' y V' en el Teorema Fundamental de Karoubi: para $R \in Alg_\ell^*$ unital hay un elemento $\theta_0 \in K_2^h((U'^2)R)$ cuyo producto cup induce un isomorfismo

$${}_\varepsilon K_*^h(V'(R)) \cong -{}_\varepsilon K_{*+1}^h(U'(R)).$$

Probamos una adjunción entre kk^h y la *K*-teoría algebraica bivariante kk definida por Cortiñas y Thom y la usamos para probar una versión del teorema de Karoubi en kk^h : el producto con la imagen de θ_0 en $KH_0^h(U^2\ell)$ induce un isomorfismo en kk^h

$$j^h({}_\varepsilon V A) \cong \Omega j^h(-{}_\varepsilon U A)$$

para todo $A \in Alg_\ell^*$. Esto nos permite obtener una versión bivariante homotópica de la clásica sucesión de 12 términos de Karoubi para la *K*-teoría hermitiana.

Palabras clave: *K*-teoría hermitiana algebraica, teorema fundamental de Karoubi, *K*-teoría hermitiana homotópica, *K*-teoría algebraica bivariante, grupos bivariantes de Witt

Acknowledgements

Introduction

Since the introduction of Kasparov’s bivariant K -theory for C^* -algebras KK [Kas80], Higson’s theorem on the universality of KK [Hig87] and Cuntz’s foundational work [Cun87; Cun05], the development of bivariant versions of K -theory has been useful and important in many computations. This ranges from applications to the Baum-Connes conjecture, classification theory of C^* -algebras such as the Elliott program and the Kirchberg-Philips theorem but also to put some constructions in different versions of K -theory — between different topological versions such as C^* -algebras, Banach $(*)$ -algebras and bornological algebras and also algebraic K -theory — on common ground. It also has been very fruitful in proving some cases of the Baum-Connes conjecture.

Cortiñas and Thom developed in [CT07] a bivariant version of algebraic K -theory with many similarities to KK , adapting them to an algebraic setting. Let ℓ be a commutative ring and write Alg_ℓ as the category of (associative) algebras over ℓ . Also fix an underlying category \mathfrak{U} for Alg_ℓ such as that of sets or that of ℓ -modules and a forgetful functor $F : Alg_\ell \rightarrow \mathfrak{U}$. Cortiñas and Thom construct a triangulated category kk which has the same objects as Alg_ℓ together with a functor $j : Alg_\ell \rightarrow kk$ which is the identity on objects and satisfies:

- Matrix stability: the natural inclusion of $A \hookrightarrow M_\infty A$ on the upper left corner maps to an isomorphism through j .
- Polynomial homotopy invariance: the inclusion $A \rightarrow A[t]$ as constants maps to an isomorphism through j .
- The functor j is an excisive homology theory for extensions which are split in \mathfrak{U} , that is, for an extension

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in Alg_ℓ which has a section $F(C) \rightarrow F(B)$, there is a natural (with respect to extensions) map $\partial : \Omega j(C) \rightarrow j(A)$ such that

$$\Omega j(C) \rightarrow j(A) \rightarrow j(B) \rightarrow j(C)$$

is a triangle in kk .

Moreover, for any triangulated category \mathfrak{T} and functor $H : Alg_\ell \rightarrow \mathfrak{T}$ which satisfies the above mentioned properties, there is a unique triangulated functor $\overline{H} : kk \rightarrow \mathfrak{T}$

such that $H = \overline{H} \circ j$. A very important property of kk is that it recovers Weibel's homotopy K -theory as a representable functor

$$\mathrm{hom}_{kk}(\ell, A) = KH_0(A).$$

There have been alternative constructions of kk by Garkusha — who also constructed bivarant K -theory versions without matrix stability — [Gar13; Gar14; Gar16] and Rodríguez Cirone [Rod20]. Also, there have been generalizations of the original construction of kk to algebras with an action of a group and group graded algebras [Ell14] and to algebras with quantum group actions [Ell18].

In this thesis we construct a generalization of kk which incorporates algebras with involution: for a ring R , an involution is a ring morphism $(-)^* : R \rightarrow R^{op}$ with $(r^*)^* = r$. Suppose now that ℓ has an involution and an element λ which satisfies

$$\lambda + \lambda^* = 1. \tag{Intro.1}$$

Consider the category Alg_ℓ^* of ℓ -algebras with involution compatible with the involution of ℓ .

Let $R \in Alg_\ell^*$ unital and $\varepsilon \in R$ central unitary (i.e. $\varepsilon^{-1} = \varepsilon^*$). An element $\phi \in R$ is called ε -hermitian if $\phi^* = \varepsilon\phi$. For an invertible ε -hermitian element, we define R^ϕ as the $*$ -ring which is the same as R as rings but with involution

$$r^\phi = \phi^{-1}r^*\phi.$$

When $A \trianglelefteq R$ is a $*$ -ideal, this involution restricts to a new involution in A and we also write A^ϕ for A equipped with involution. We say a functor $H : Alg_\ell^* \rightarrow \mathfrak{C}$ is *hermitian stable* if for any $R \in Alg_\ell^*$ unital and $A \trianglelefteq R$ and invertible ε -hermitian elements $\phi, \psi \in R$ the inclusion on the upper left corner

$$i_\phi : A^\phi \rightarrow M_2(A)^{\phi \oplus \psi}$$

is mapped to an isomorphism through H .

In Chapter 3 we construct a triangulated category kk^h which has the same objects as Alg_ℓ^* together with a functor $j^h : Alg_\ell^* \rightarrow kk^h$ which is the identity on objects. One of the key pieces in this construction is the ability to fix some standard polynomial homotopies commonly occurring on K -theory (such as rotation homotopies) which are not involution preserving; this is mainly fixed with Lemma 1.2.3; the existence of the element (Intro.1) is essential. The main result in Chapter 3 is the following:

Theorem (Theorem 3.2.17 and Theorem 3.2.20) *There is a triangulated category kk^h and an excisive homology theory functor $j^h : Alg_\ell^* \rightarrow kk^h$ which is matricially and hermitian stable and polynomial homotopy invariant.*

Furthermore, the functor $j^h : Alg_\ell^ \rightarrow kk^h$ is universal between the matricially and hermitian stable, polynomial homotopy invariant excisive homology theories.*

For a unital ring with involution R and a central unitary element $\varepsilon \in R$, recall the hermitian algebraic K -theory spectra ${}_e K^h(R)$ as defined in [Lod76]. In Chapter

2 we define a Weibel style homotopy invariant version of ${}_{\varepsilon}K_*^h(R)$ which we denote ${}_{\varepsilon}KH_*^h(R)$.

In Chapter 4 we discuss some standard computations such as classification of the image through j^h of coproducts, the Toeplitz algebra, and the Cohn algebra of a finite graph and also prove the algebraic analogue of the Pimsner-Voiculescu sequence. We also show the following result:

Theorem(Theorem 4.2.1) There is a natural isomorphism

$$\mathrm{hom}_{kk^h}(\ell, A) \cong KH_0^h(A). \quad (\text{Intro.2})$$

For a unital $*$ -ring R , there are natural maps between the K -theory spectra and the hermitian K -theory spectra induced by the hyperbolic and the forgetful maps

$$\mathrm{hyp} : K(R) \rightarrow {}_{\varepsilon}K^h(R) \quad \mathrm{forg} : {}_{\varepsilon}K^h(R) \rightarrow K(R)$$

Write ${}_{\varepsilon}\mathcal{U}(R)$ and ${}_{\varepsilon}\mathcal{V}(R)$ for the homotopy fibers of these maps. Assume that R has an element as in (Intro.1). Karoubi's Fundamental Theorem for hermitian K -theory [Kar80] shows that there are natural homotopy equivalences

$${}_{\varepsilon}\mathcal{V}(R) \sim \Omega_{-{\varepsilon}}\mathcal{U}(R).$$

Moreover, Karoubi constructs functors U', V' for rings with involutions such that there are homotopy equivalences

$${}_{\varepsilon}K^h(U'R) \sim {}_{\varepsilon}\mathcal{U}(R) \text{ and } {}_{\varepsilon}K^h(V'R) \sim {}_{\varepsilon}\mathcal{V}(R).$$

Karoubi also shows that there is a natural equivalence

$${}_{\varepsilon}K^h(U'V'R) \sim {}_{-{\varepsilon}}K^h(R).$$

Thus, we can rephrase Karoubi's fundamental theorem as the equivalence

$${}_{\varepsilon}K^h(R) \sim \Omega_{-{\varepsilon}}^2 K^h((U')^2 R). \quad (\text{Intro.3})$$

The equivalence (Intro.3) is induced by the cup product with an element in $\theta_0 \in {}_{-1}K_2^h((U')^2(\mathbb{Z}))$.

In Chapter 5 we show that kk^h has a adjunction with kk which is analogue to the maps hyp and forg in the homotopy invariant setting. Then we construct functors ${}_{\varepsilon}U, {}_{\varepsilon}V : \mathrm{Alg}_{\ell}^* \rightarrow \mathrm{Alg}_{\ell}^*$ such that composing with the functor of homotopy hermitian algebraic K -theory we recover the homotopy versions of ${}_{\varepsilon}\mathcal{V}$ and ${}_{\varepsilon}\mathcal{U}$ up to a degree shift. Using the aforementioned adjunction we show that ${}_{\varepsilon}U$ and ${}_{\varepsilon}V$ have analogue properties in kk^h to those of U' and V' for hermitian K -theory. Write θ for the image of θ_0 in ${}_{-1}KH_0^h(U^2\ell)$. The main result of Chapter 5 is

Theorem (Theorem 5.3.1 and Corollary 5.3.2) The product with θ induces for every $A \in \mathrm{Alg}_{\ell}^*$ an isomorphism in kk^h

$$j^h(A) \cong j^h({}_{-1}U^2(A)), \quad (\text{Intro.4})$$

which gives an isomorphism in kk^h

$$j^h({}_\varepsilon VA) \cong j^h({}_{-\varepsilon}UA).$$

Let R be a unital $*$ -ring with an element λ which satisfies (Intro.1). The involution of R induces an involution $g \rightarrow (g^*)^{-1}$ in $\mathrm{GL}_\infty(R)$ which in turn induces a natural action of $\mathbb{Z}/2$ in $K_*(R)$; for $x \in K_n(R)$ write \bar{x} for this action. Recall the Witt and coWitt groups ${}_\varepsilon W_n(R)$ and ${}_\varepsilon W'_n(R)$ and write $k_n(R)$ and $k'_n(R)$ for the $\mathbb{Z}/2$ -Tate cohomology groups of $K_n(R)$ with the aforementioned action. Using the equivalence (Intro.3), Karoubi shows that there is a 12-term exact sequence:

$$\begin{array}{cccccccccccc} k_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W_{n+2}(R) & \longrightarrow & {}_\varepsilon W'_n(R) & \longrightarrow & k'_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W'_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W_n(R) \\ \uparrow & & & & & & & & & & \downarrow \\ W_{n+1}(R) & \longleftarrow & W'_{n+1}(R) & \longleftarrow & k'_{n+1}(R) & \longleftarrow & {}_{-\varepsilon}W'_n(R) & \longleftarrow & W_{n+2}(R) & \longleftarrow & k_{n+1}(R) \end{array}$$

In the end of Chapter 5, we show that for bivariant adaptation of these groups (Definition 5.3.6) and we have a 12-term exact sequence (Theorem 5.3.7):

$$\begin{array}{cccccccccccc} k_{n+1}(A, B) & \longrightarrow & {}_{-\varepsilon}W_{n+2}(A, B) & \longrightarrow & {}_\varepsilon W'_n(A, B) & \longrightarrow & k'_{n+1}(A, B) & \longrightarrow & {}_{-\varepsilon}W'_{n+1}(A, B) & \longrightarrow & {}_{-\varepsilon}W_{n+1}(A, B) \\ \uparrow & & & & & & & & & & \downarrow \\ {}_\varepsilon W_{n+1}(A, B) & \longleftarrow & {}_\varepsilon W'_{n+1}(A, B) & \longleftarrow & k'_{n+1}(A, B) & \longleftarrow & {}_{-\varepsilon}W'_n(A, B) & \longleftarrow & {}_\varepsilon W_{n+2}(A, B) & \longleftarrow & k_n(A, B) \end{array}$$

The rest of this thesis is outlined the following way. In Chapter 1 we discuss preliminary concepts and prove some useful lemmas that we will use throughout the thesis. In Chapter 2 we recall the construction of hermitian K -theory, we define KH^h and prove some of its basic properties; we also discuss the product structure of K^h and how it passes to KH^h . We end the chapter recalling Karoubi's Fundamental Theorem for hermitian algebraic K -theory. In Chapter 3 we construct the category kk^h and the functor $j^h : \mathrm{Alg}_\ell^* \rightarrow kk^h$; first we prove the necessary technical lemmas to construct the morphism sets and then we show some of its properties as a triangulated category and how j^h is a universal excisive homology theory with matrix and hermitian stability and homotopy invariance. In Chapter 4 we proceed to develop some computations as a matter of examples and show (Intro.2). In Chapter 5 we show the adjunction between kk^h and kk and construct the functors U, V ; we prove some of their properties in order to show (Intro.4) and obtain the 12-term exact sequence from it.

Chapter 1

Preliminaries

1.1 Rings and algebras with involution

Fix a commutative ring ℓ . An ℓ -algebra is a ring A together with a symmetric ℓ -module structure such that the product is ℓ -bilinear.

Suppose ℓ has an involution: a ring isomorphism $*$: $\ell \rightarrow \ell^{op} = \ell$, such that $(x^*)^* = x$, for all $x \in \ell$. A $*$ -algebra over ℓ , is an ℓ -algebra A together with an involution $*$: $A \rightarrow A^{op}$ that is semilinear with respect to the module action:

$$(xa)^* = x^*a^* \text{ for } x \in \ell \text{ and } a \in A.$$

An ℓ -algebra morphism is a ring morphism that is also an ℓ -bimodule morphism. We write Alg_ℓ for the category of ℓ -algebras with ℓ -algebra morphisms and Alg_ℓ^* for the category of $*$ -algebras over ℓ with $*$ -morphisms, that is, ℓ -algebra morphisms that preserve the involution. A $*$ -ideal in a $*$ -algebra is a two-sided ideal that is closed under the action of ℓ and under the involution. For a $*$ -ideal $I \trianglelefteq A$, the quotient A/I is also a $*$ -algebra with the induced involution.

Example 1.1.1. For any commutative ring ℓ , the identity map $\text{id} : \ell \rightarrow \ell$ is an involution; it is called the trivial involution. In the case of $\ell = \mathbb{Z}$ it is the only involution and $Alg_{\mathbb{Z}} = Rings$ is the category of rings; the category $Rings^* = Alg_{\mathbb{Z}}^*$ is called the category of $*$ -rings.

Example 1.1.2. Let A and B be $*$ -algebras over ℓ . The tensor product $A \otimes_\ell B$ is a $*$ -algebra over ℓ with involution $(a \otimes b)^* = a^* \otimes b^*$. In some cases we write LA for $L \otimes_\ell A$ and write $L : Alg_\ell^* \rightarrow Alg_\ell^*$ for the functor given by tensoring with L . Except when explicitly noted, all tensor products will be over ℓ .

Example 1.1.3. Write M_n for the ring of $n \times n$ matrices over ℓ . The ℓ -algebra M_n has a natural involution $(a_{ij})^* = a_{ji}^*$.

More generally, let X be a set and define

$$\Gamma_X = \{a : X \times X \rightarrow \ell : \text{im}(a) \text{ is finite and} \\ \exists N \text{ s.t. } \forall x \in X |\{y \in X : a(x, y) \neq 0\}|, |\{y \in X : a(y, x) \neq 0\}| \leq N\}.$$

with convolution product and conjugate transposition

$$(ab)(x, y) = \sum_{z \in X} a(x, z)b(z, y),$$

$$a^*(x, y) = a(y, x)^*$$

make Γ_X a $*$ -algebra over ℓ . We write $M_X \trianglelefteq \Gamma_X$ for the $*$ -ideal of finitely supported functions and Σ_X for the quotient Γ_X/M_X . We also write $\Gamma = \Gamma_{\mathbb{N}}$, $M_{\infty} = M_{\mathbb{N}}$ and $\Sigma = \Sigma_{\mathbb{N}}$. When X has cardinality n then $M_n \cong M_X = \Gamma_X$. For a $*$ -algebra A we write $\Gamma_X A$, $M_X A$ and $\Sigma_X A$ for the tensor product of Γ_X , M_X and Σ_X with A respectively as in Example 1.1.2. We also write Σ_X^n for $\Sigma_X^{\otimes n}$

Example 1.1.4 (Unitalization). Let A be a $*$ -algebra and define $\tilde{A} = A \oplus \ell$ as an ℓ -bimodule with the following multiplication and involution

$$(a, x)(b, y) = (ab + ay + xb, xy)$$

$$(a, x)^* = (a^*, x^*).$$

The $*$ -algebra \tilde{A} is unital and has a natural morphism $A \rightarrow \tilde{A}$, $a \mapsto (a, 0)$ which maps A isomorphically to an ideal in \tilde{A} . The quotient \tilde{A}/A is isomorphic to ℓ and the quotient map $\tilde{A} \rightarrow \ell$ is split by $x \mapsto (0, x)$; whenever A is unital the unitalization \tilde{A} is isomorphic to $A \times \ell$ by means of this splitting.

Example 1.1.5 (Amalgamated coproducts and sums). Let $A, B, C \in \text{Alg}_{\ell}^*$ and $i : C \rightarrow A$ and $j : C \rightarrow B$ two $*$ -morphisms with retractions $\alpha : A \rightarrow C$ and $\beta : B \rightarrow C$ (i.e. $\alpha i = \text{id}_C$ and $\beta j = \text{id}_C$). The amalgamated coproduct of A and B over C is the ℓ -module

$$A \amalg_C B := C \oplus \ker \alpha \oplus \ker \beta \oplus (\ker \alpha \otimes_{\tilde{C}} \ker \beta) \oplus (\ker \beta \otimes_{\tilde{C}} \ker \alpha) \oplus \dots$$

Where each summand beyond the first is given by the tensor product of $\ker \alpha$ and $\ker \beta$ in all possible orderings with an increasing number of tensor factors. This defines an ℓ -algebra with product given by concatenation of elementary tensors and extended by bilinearity. It also has an involution given by the involutions of A, B and C and twisting the elementary tensors appropriately. In the case $C = 0$, we write $A \amalg B$; this is simply the coproduct of A and B as ℓ -algebras.

The direct sum all tensors with two or more factors forms an ideal $K \trianglelefteq A \amalg_C B$ and we define the amalgamated direct sum as the quotient

$$A \oplus_C B := A \amalg_C B / K.$$

When $A = B$ and $C = 0$ we write $Q(A) := A \amalg A$ and $\iota_0, \iota_1 : A \rightarrow Q(A)$ for the natural inclusions of A . The identity of $\text{id}_A : A \rightarrow A$ induces a $*$ -morphism $\text{id}_A \amalg \text{id}_A : Q(A) \rightarrow A$ and we write $q(A)$ for the kernel of this map. There are also two natural maps $\pi_0, \pi_1 : q(A) \rightarrow A$ which are the restrictions of $\text{id}_A \amalg 0$ and $0 \amalg \text{id}_A$ to $q(A)$.

Example 1.1.6 (Free involutions and induction). Let A be a ring. Define $\text{inv}(A) = A \oplus A^{\text{op}}$ with involution $(a, b)^* = (b, a)$. This gives rise to an equivalence

$$\text{inv} : \text{Alg}_\ell \rightarrow \text{Alg}_{\text{inv}(\ell)}^*$$

with inverse $A \mapsto (1, 0)A$. There is a natural $*$ -morphism $\eta : \ell \rightarrow \text{inv}(\ell)$ defined by $\eta(x) = (x, x^*)$. We can restrict the action of an $\text{inv}(\ell)$ -algebra to ℓ through η . Composing the functor inv with the restriction of scalars gives rise to a functor

$$\text{ind} : \text{Alg}_\ell \rightarrow \text{Alg}_\ell^*.$$

This functor is right adjoint to the forgetful functor $\text{res} : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell$ with unit and counit given by

$$\begin{aligned} \eta_A : A &\rightarrow \text{ind}(\text{res}(A)) = A \oplus A^{\text{op}} & (1.1.7) \\ a &\mapsto (a, a^*) \text{ and} \end{aligned}$$

$$\begin{aligned} \text{pr}_1 : \text{res}(\text{ind}(B)) = B \oplus B^{\text{op}} &\rightarrow B & (1.1.8) \\ (x, y) &\mapsto x \end{aligned}$$

respectively.

Similarly, for an ℓ -algebra A define $\text{ind}'(A) := A \amalg A^{\text{op}}$ with involution which permutes the copies of A . This gives a functor $\text{ind}' : \text{Alg}_\ell \rightarrow \text{Alg}_\ell^*$ which is left adjoint to $\text{res} : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell$ with unit and counit given by

$$\begin{aligned} \tilde{\eta}_A : A &\rightarrow \text{res}(\text{ind}'(A)) = A \amalg A^{\text{op}} & (1.1.9) \\ a &\mapsto \iota_0(a) \text{ and} \end{aligned}$$

$$\begin{aligned} \text{id}_B \amalg \text{id}_B^* : \text{ind}'(\text{res}(B)) = B \amalg B^{\text{op}} &\rightarrow B & (1.1.10) \\ i_0(b) &\mapsto b \\ i_1(b) &\mapsto b^* \end{aligned}$$

respectively.

Definition 1.1.11 (Hermitian elements and involutions). Let R be a unital ring with involution and $\varepsilon \in R$. We say that ε is *unitary* if it is invertible and $\varepsilon^* = \varepsilon^{-1}$ (e.g. $\varepsilon = \pm 1$).

For $\varepsilon \in R$ central unitary and $\phi \in R$, we say that ϕ is ε -*hermitian* if $\phi = \varepsilon\phi^*$. If $\phi \in R$ is invertible and ε -hermitian then we can define a new involution in R by

$$r \mapsto r^\phi := \phi^{-1}r^*\phi.$$

We write R^ϕ for the ring R with this new involution. If S is another unital $*$ -algebra over ℓ and ψ is η -hermitian and invertible then $\phi \otimes \psi \in R \otimes_\ell S$ is $\varepsilon \otimes \eta$ -hermitian and invertible and

$$(R \otimes_\ell S)^{\phi \otimes \psi} = R^\phi \otimes S^\psi. \quad (1.1.12)$$

Remark 1.1.13. Let R be a unital ring, $A \trianglelefteq R$ a $*$ -ideal, $\varepsilon \in R$ central unitary and $\phi \in R$ an invertible ε -hermitian. The involution defined in Definition 1.1.11 restricts properly to an involution on A and we write A^ϕ for A equipped with this new involution.

Definition 1.1.14. Let A be a ring with involution and $u \in A$ unitary. The map

$$\begin{aligned} \text{ad}(u) : A &\rightarrow A \\ x &\mapsto uxu^{-1} \end{aligned}$$

defines a $*$ -isomorphism with inverse $\text{ad}(u^*)$.

Remark 1.1.15. Let R be a unital $*$ -algebra over ℓ , $\varepsilon \in R$ central unitary and $\phi, \psi \in R$ invertible ε -hermitian. If there exists $u \in R$ invertible such that $\psi = u^*\phi u$ then $\text{ad}(u) : R^\psi \rightarrow R^\phi$ is a $*$ -isomorphism.

Example 1.1.16. Let R_0 be an ℓ -algebra and $R = \text{inv}(R_0) \in \text{Alg}_\ell^*$. If $\varepsilon = (\varepsilon_0, \varepsilon_1) \in R$ is central unitary then ε_0 and ε_1 are central and

$$\begin{aligned} (1, 1) &= (\varepsilon_0, \varepsilon_1)^*(\varepsilon_0, \varepsilon_1) \\ &= (\varepsilon_1, \varepsilon_0)(\varepsilon_0, \varepsilon_1) \\ &= (\varepsilon_1\varepsilon_0, \varepsilon_0\varepsilon_1); \end{aligned}$$

therefore, $\varepsilon_1 = \varepsilon_0^{-1}$. We can deduce from this that any invertible ε -hermitian element $\phi \in R$ is of the form

$$\phi = (\phi_0, \varepsilon_0^{-1}\phi_0) = (1, \phi_0)^*(1, \varepsilon_0^{-1})(1, \phi_0).$$

It follows from Remark 1.1.15 that $R^\phi \cong R^{(1, \varepsilon_0^{-1})} = R$ since ε_0 is central.

Example 1.1.17. Let P be a finitely generated projective ℓ -module. An ε -hermitian bilinear form is a map $\psi : P \times P \rightarrow \ell$ which is ℓ -linear in the first coordinate and satisfies

$$\psi(x, y) = \varepsilon\psi(y, x)^*.$$

We say that ψ is non-degenerate if $\psi(-, y) : P \rightarrow P^*$ is an isomorphism for all $y \in P$; in this case we say that the pair (P, ψ) is an ε -hermitian module.

For an ε -hermitian module (P, ψ) , the non-degeneracy of ψ induces an involution on the ℓ -algebra of ℓ -linear endomorphisms $\text{End}(P)$. This involution is determined by the following property: for $T \in \text{End}(P)$ and $x, y \in P$ we have

$$\psi(T(x), y) = \psi(x, T^*(y)).$$

If $P = \ell^n$ is free, then $\text{End}(P) \cong M_n$ and the involution induced by the ε -hermitian form ψ , corresponds to an ε -hermitian invertible h_ψ and the involution $(-)^{h_\psi}$.

Example 1.1.18. Consider the invertible -1 -hermitian element

$$h_{\pm} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2.$$

We write $M_{\pm} = (M_2)^{h_{\pm}}$ as in Example 1.1.11 and $M_{\pm}A$ for $M_{\pm} \otimes A$. We write $i_+, i_- : \ell \rightarrow M_{\pm}$ for the $*$ -morphisms defined by the upper left and lower right corner inclusions respectively.

The element h_{\pm} corresponds to the *hyperbolic* hermitian module: for $H(\ell) = \ell^2$, the -1 -hermitian form

$$h((x_1, y_1), (x_2, y_2)) = x_1y_2 - x_2y_1.$$

It is well known [see for example KV73, Theorem 1.4] that for any hermitian module (P, ψ) then $(P, -\psi)$ is also a hermitian module and

$$(P, \psi) \oplus (P, -\psi) \cong H(\ell) \otimes P.$$

in such a way that the bilinear forms are preserved through this isomorphism.

Similarly, let $\varepsilon \in \ell$ be central unitary and consider the invertible ε -hermitian element

$$h_{\varepsilon} = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \in M_2.$$

We write ${}_{\varepsilon}M_2 = (M_2)^{h_{\varepsilon}}$ and ${}_{\varepsilon}M_2A$ for ${}_{\varepsilon}M_2 \otimes A$.

Due to (1.1.12) we have the identity

$${}_{\varepsilon}M_2 {}_{\eta}M_2 \cong {}_{\varepsilon\eta}M_2 M_2. \quad (1.1.19)$$

Let X be an infinite set and fix a bijection $\{1, 2\} \times X \cong X$. This bijection together with (1.1.19) induces $*$ -isomorphisms

$${}_{\eta}M_2 {}_{\varepsilon}M_2 M_X \cong {}_{\eta\varepsilon}M_2 M_{\{1,2\} \times X} \cong {}_{\eta\varepsilon}M_2 M_X. \quad (1.1.20)$$

Example 1.1.21 (Polynomial $*$ -algebras). We consider the polynomial ring $\ell[t]$ with the involution which fixes t . For any 1-hermitian element $\alpha \in A$ the evaluation map $\text{ev}_{\alpha} : \ell[t] \rightarrow \ell$ that maps $t \mapsto \alpha$ is a $*$ -morphism.

We write

$$\begin{aligned} P &= \ker(\text{ev}_0 : \ell[t] \rightarrow \ell) \text{ and} \\ \Omega &= \ker(\text{ev}_1 : P \rightarrow \ell). \end{aligned}$$

for the *path* and *loop* algebras respectively. We also consider the Laurent polynomial algebra $\ell[t, t^{-1}]$ with involution that interchanges t and t^{-1} , $t^* = t^{-1}$. For any unitary element $u \in \ell$ we have an evaluation map $\text{ev}_u : \ell[t, t^{-1}] \rightarrow \ell$ which maps $t \mapsto u$.

As with matrices we write $A[t]$, $A[t, t^{-1}]$, PA and ΩA for $\ell[t] \otimes_{\ell} A$, $\ell[t, t^{-1}] \otimes_{\ell} A$, $P \otimes_{\ell} A$ and $\Omega \otimes A$ respectively. We write Ω^n for $\Omega^{\otimes n}$.

Example 1.1.22 (Simplicial $*$ -algebras). Let $n \in \mathbb{N}_0$ and

$$\ell[t_1, \dots, t_n] = \ell[t_1] \otimes \cdots \otimes \ell[t_n]$$

be the polynomial algebra in n variables. We define

$$\ell^{\Delta^n} := \ell[t_0, \dots, t_n] / \langle t_0 + \cdots + t_n - 1 \rangle.$$

This defines a simplicial $*$ -algebra

$$\begin{aligned} \ell^\Delta : \Delta^{op} &\rightarrow \text{Alg}_\ell^* \\ [n] &\mapsto \ell^{\Delta^n}, \end{aligned}$$

and we write A^Δ for $\ell^\Delta \otimes A$. Write \mathfrak{S} for the category of simplicial sets. Let $X \in \mathfrak{S}$ and $B_\bullet : \Delta^{op} \rightarrow \text{Alg}_\ell^*$ be a simplicial $*$ -algebra. The set $\text{hom}_{\mathfrak{S}}(X, B_\bullet)$ is a $*$ -algebra. For $X \in \mathfrak{S}$ and $A \in \text{Alg}_\ell^*$ we define the $*$ -algebra of functions on the simplicial set X as

$$A^X := \text{hom}_{\mathfrak{S}}(X, A^\Delta).$$

A pointed simplicial set (X, x) is a simplicial set X together with a map $x : \text{pt} = \Delta^0 \rightarrow X$. Write $\text{ev}_x : A^X \rightarrow A^{\text{pt}}$ for the induced $*$ -morphism and define

$$A^{(X, x)} := \ker(\text{ev}_x).$$

Remark 1.1.23. Some of the $*$ -algebras mentioned in Example 1.1.21 are particular cases of Example 1.1.22:

$$\begin{aligned} A^{\Delta^1} &\cong A[t], \\ A^{(\Delta^1, \text{pt})} &\cong PA \end{aligned}$$

and writing $S^1 = \Delta^1 / \Delta^0$ for the simplicial circle,

$$A^{(S^1, \text{pt})} \cong \Omega A.$$

Throughout this thesis, we will often assume the following:

λ -assumption 1.1.24. the ring contains an element λ such that $\lambda + \lambda^* = 1$.

Example 1.1.25. The λ -assumption 1.1.24 is satisfied for example when 2 is invertible in putting $\lambda = 1/2$. Another example is when $\ell = \text{inv}(\ell_0)$ for some ring ℓ_0 and $\lambda = (1, 0)$.

Remark 1.1.26. Suppose that ℓ satisfies the λ -assumption 1.1.24 and let $\varepsilon \in \ell$ be unitary, R be a unital $*$ -algebra and $\phi \in R$ be an invertible ε -hermitian element. Recall the matrices h_\pm and h_ε from Example 1.1.18. The matrix

$$u_\lambda = \begin{pmatrix} 1 & 1 \\ \lambda\phi^* & -\lambda^*\phi^* \end{pmatrix} \tag{1.1.27}$$

satisfies $u_\lambda^*(h_\varepsilon \otimes 1)u_\lambda = h_\pm \otimes \phi$, whence $\text{ad}(u_\lambda) : M_\pm R^\phi \rightarrow {}_\varepsilon M_2 R$ is a $*$ -isomorphism. Taking $R = \ell$ and $\varepsilon = \phi = 1$ we get $M_\pm \cong {}_1 M_2$.

1.2 Algebraic homotopies

Definition 1.2.1. Let $A, B \in \text{Alg}_\ell^*$ and $f, g : A \rightarrow B$ two $*$ -morphisms. We say that f and g are *elementary (algebraically) $*$ -homotopic* if there exists a $*$ -morphism $H : A \rightarrow B[t]$, called a $*$ -homotopy, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{H} & B[t] \\ & \searrow (f,g) & \downarrow (\text{ev}_0, \text{ev}_1) \\ & & B \times B \end{array}$$

commutes. We say that f, g are *(algebraically) $*$ -homotopic* if there exists a finite sequence $f_0, \dots, f_n : A \rightarrow B$ of $*$ -morphisms such that $f_0 = f$, $f_n = g$ and f_i is elementary $*$ -homotopic to f_{i+1} for $i = 0, \dots, n-1$; whenever f and g are $*$ -homotopic we write $f \sim^* g$.

It is immediate from this definition that homotopy is an equivalence relation that is compatible with composition of $*$ -morphisms. We write $[A, B]$ for set of equivalence classes of $*$ -morphisms $A \rightarrow B$ modulo homotopy. The sets $[-, -]$ have a composition law and therefore are the arrows of a category $[\text{Alg}_\ell^*]$ which has $*$ -algebras as objects.

Definition 1.2.2. Let $F : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ be a functor. We say that F is *homotopy invariant* if $F(f) = F(g)$ whenever $f \sim^* g$.

Let $C \in \text{Alg}_\ell$ and $A, B \subseteq C$ subalgebras. Suppose $u, v \in C$ satisfy

$$\begin{aligned} uAv &\subseteq B \text{ and} \\ avua' &= aa' \text{ for all } a, a' \in A. \end{aligned}$$

Then

$$\begin{aligned} \text{ad}(u, v) : A &\rightarrow B \\ a &\mapsto uav \end{aligned}$$

is an algebra morphism. We say that the pair (u, v) *multiplies A into B* . Let $u_0, u_1, v_0, v_1 \in C$ such that (u_0, v_0) and (u_1, v_1) multiplies A into B . A *homotopy between the pairs (u_0, v_0) and (u_1, v_1)* is a pair $(u(t), v(t)) \in C[t]^2$ that multiplies A (as constants in $C[t]$) into $B[t]$ and that $(u(i), v(i)) = (u_i, v_i)$ (for $i = 0, 1$). In this case $\text{ad}(u(t), v(t)) : A \rightarrow B[t]$ is a homotopy between $\text{ad}(u_0, v_0)$ and $\text{ad}(u_1, v_1)$. Suppose now that C is a $*$ -algebra and that A, B are $*$ -subalgebras; when $v = u^*$ and the pair (u, u^*) multiplies A into B , we have that $\text{ad}(u, u^*)$ is a $*$ -morphism. In this case we say that u *$*$ -multiplies A into B* . If $u, w \in C$ both $*$ -multiply A into B , a $*$ -homotopy between u and w is an element $z(t) \in C[t]$ $*$ -multiplying A into $B[t]$ such that $z(0) = u$ and $z(1) = w$. We shall often encounter examples of elements $u_0, u_1 \in C$ which $*$ -multiply A into B that are homotopic via a pair $(u(t), v(t))$ with $u(t)^* \neq v(t)$ so that the homotopy $\text{ad}(u(t), v(t))$ is not a $*$ -morphism. This can be fixed as follows.

Lemma 1.2.3. *Suppose ℓ satisfies the λ -assumption 1.1.24. Let $C \in \text{Alg}_\ell^*$, $A, B \subseteq C$ $*$ -subalgebras, $u_0, u_1 \in C$ that $*$ -multiply A into B and $(v, w) \in C[t]^2$ a homotopy between (u_0, u_0^*) and (u_1, u_1^*) . Assume as well that*

$$w^*Aw, vAv^* \subseteq B[t].$$

Then

$$c(v, w) = \begin{pmatrix} \lambda^*v + \lambda w^* & \lambda^*(v - w^*) \\ \lambda(v - w^*) & \lambda v + \lambda^*w^* \end{pmatrix} \in M_\pm C[t]$$

$*$ -multiplies $i_+(A)$ into $M_\pm B[t]$ and $\text{ad}(c(u, v), c(u, v)^*) \circ i_+$ is a $*$ -homotopy between $i_+ \text{ad}(u_0, u_0^*)$ and $i_+ \text{ad}(u_1, u_1^*)$.

Proof. A straightforward computation shows that

$$c(v, w)^*c(v, w) = c(wv, wv).$$

Hence, for $a, a' \in A$ we have

$$\begin{aligned} i_+(a)c(v, w)^*c(v, w)i_+(a') &= i_+(a)c(wv, wv)i_+(a') \\ &= i_+(a(\lambda^*wv + \lambda(wv)^*))a' \\ &= i_+(\lambda^*awva' + \lambda a(wv)^*a') \\ &= i_+(\lambda^*aa' + \lambda(a'^*wva^*)^*) \\ &= i_+(aa'(\lambda^* + \lambda)) \\ &= i_+(aa'). \end{aligned}$$

Similarly, $c(v, w)i_+(A)c(v, w)^* \subseteq M_\pm B[t]$. Thus, $H = \text{ad}(c(u, v))i_+ : A \rightarrow M_\pm B[t]$ is a $*$ -morphism and for $i = 0, 1$ we get

$$\text{ev}_i(c(u, v)) = c(u_i, u_i) = \begin{pmatrix} u_i & 0 \\ 0 & u_i \end{pmatrix},$$

so that $\text{ev}_i H = i_+ \text{ad}(u_i, u_i^*)$. □

Definition 1.2.4. Let $p, q \geq 0$ and $n = p + q$. Define

$$i_+^{p,q} := (M_\pm)^{\otimes p} \otimes i_+ \otimes (M_\pm)^{\otimes q} : M_\pm^{\otimes n} \rightarrow M_\pm^{\otimes n+1}$$

Lemma 1.2.5. *Let p, q and n be as above, and let $p', q' \geq 0$ be such that $p' + q' = n + 1$. Then $i_+^{p',q'} i_+^{p,q}$ is $*$ -homotopic to $i_+^{0,n+1} i_+^{0,n}$.*

Proof. First observe that we have $i_+^{0,0} = i_+$ and $i_+^{1,0} i_+ = i_+^{0,1} i_+$. Therefore, tensoring with identity maps we get

$$i_+^{r,s+1} i_+^{r,s} = i_+^{r+1,s} i_+^{r,s} \tag{1.2.6}$$

for any $r, s \geq 0$. Next, under the identification $M_2 \otimes M_2 = M_{\{1,2\}^2}$, we have $i_+^{1,0}(e_{i,j}) = e_{(i,1),(j,1)}$ and $i_+^{0,1}(e_{i,j}) = e_{(1,i),(1,j)}$. One checks that the matrix

$$u = e_{(1,1),(1,1)} - e_{(1,2),(2,1)} + e_{(2,1),(1,2)} + e_{(2,2),(2,2)}$$

is a unitary element of $M_{\pm}^{\otimes 2}$ and satisfies $\text{ad}(u)i_{+}^{1,0} = i_{+}^{0,1}$. Moreover by [CT07, Section 6.4], there exists an invertible element $u(t) \in M_{\pm}^{\otimes 2}[t]$ such that $u(0) = 1$ and $u(1) = u$. Hence the composites of $i_{+}^{0,2}$ with $i_{+}^{1,0}$ and $i_{+}^{0,1}$ are $*$ -homotopic by Lemma 1.2.3. Tensoring on both sides with identity maps, we get that

$$i_{+}^{p,q+1}i_{+}^{p+1,q-1} \sim^* i_{+}^{p,q+1}i_{+}^{p,q}.$$

Let p', q' as in the statement. Permuting factors in the tensor product $M_{\pm}^{\otimes n+1}$ we obtain a $*$ -isomorphism $\sigma : M_{\pm}^{\otimes n+1} \rightarrow M_{\pm}^{\otimes n+1}$ such that $\sigma i_{+}^{p,q+1} = i_{+}^{p',q'}$. Hence we have

$$i_{+}^{p',q'}i_{+}^{p+1,q-1} \sim^* i_{+}^{p',q'}i_{+}^{p,q} \quad (1.2.7)$$

for all p, q, p', q' as above. The lemma follows from (1.2.7) using the identity (1.2.6). \square

1.3 Ind- $*$ -algebras

Definition 1.3.1. Let \mathfrak{C} be a category. An *ind-object* in \mathfrak{C} is a pair (C, I) consisting of an upward filtered poset I and a functor $C : I \rightarrow \mathfrak{C}$. We shall often write C_i for $C(i)$ and $(C_i)_{i \in I}$ or simply C_{\bullet} for an ind-object $C : I \rightarrow \mathfrak{C}$.

The ind-objects of a category \mathfrak{C} form a category $\text{ind} - \mathfrak{C}$ whose morphisms sets are

$$\text{hom}_{\text{ind} - \mathfrak{C}}((C_i), (D_j)) = \varprojlim_i \varinjlim_j \text{hom}_{\mathfrak{C}}(C_i, D_j).$$

Any functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ extends to $F : \text{ind} - \mathfrak{C} \rightarrow \text{ind} - \mathfrak{D}$ by applying F indexwise; $F(C)_i = F(C_i)$.

Example 1.3.2. Let $i^n : M_n \rightarrow M_{n+1}$ be upper left corner inclusion and write M_{\bullet} for the ind- $*$ -algebra

$$\begin{aligned} \mathbb{N}_0 &\rightarrow \text{Alg}_{\ell}^* \\ (n \rightarrow n+1) &\mapsto (M_n \xrightarrow{i_n} M_{n+1}). \end{aligned}$$

Similarly, recall Definition 1.2.4 and write M_{\pm}^{\bullet} for the ind- $*$ -algebra.

$$\begin{aligned} \mathbb{N}_0 &\rightarrow \text{Alg}_{\ell}^* \\ (n \rightarrow n+1) &\mapsto (M_{\pm}^n \xrightarrow{i_{+}^{0,n}} M_{\pm}^{n+1}). \end{aligned}$$

For an infinite set X we write

$$\mathcal{M}_X = M_{\pm}^{\bullet} M_X.$$

Any bijection $f : X \rightarrow Y$ induces an isomorphism $f_* : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ given by the corresponding isomorphism $M_X \cong M_Y$ and tensoring with the corresponding identities.

Example 1.3.3. For a finite simplicial set K , we write $\text{sd} K$ for the barycentric subdivision. This defines a functor $\text{sd} : \mathfrak{S} \rightarrow \mathfrak{S}$. The barycentric subdivision is equipped with a natural transformation $h : \text{sd} \rightarrow \text{id}_{\mathfrak{S}}$ so called the last vertex map [GJ99, Chapter III, Section 4, p.193]. Iterating this map, one obtains a system of simplicial sets

$$\cdots \xrightarrow{h} \text{sd}^n K \xrightarrow{h} \text{sd}^{n-1} K \rightarrow \cdots \rightarrow K.$$

Write $\text{sd}^\bullet K$ for the (contravariant) functor

$$\begin{aligned} \mathbb{N}_0 &\rightarrow s\text{Set} \\ (n \rightarrow n+1) &\mapsto (\text{sd}^{n+1} K \xrightarrow{h} \text{sd}^n K). \end{aligned}$$

For each $A \in \text{Alg}_\ell^*$ the composed functor $A^{\text{sd}^\bullet K}$ gives an ind- $*$ -algebra. This construction also applies to pointed simplicial sets in a similar way.

Some particular examples of subdivision ind- $*$ -algebras that we will use are

$$\begin{aligned} A^{\mathbb{S}^1} &= A^{\text{sd}^\bullet(S^1, \text{pt})}, \\ A^{\mathbb{S}^n} &= (A^{\mathbb{S}^{n-1}})^{\mathbb{S}^1} \text{ and} \\ \mathcal{P}A &= A^{\text{sd}^\bullet(\Delta, \text{pt})}. \end{aligned}$$

Remark 1.3.4. The two endpoint inclusions $\Delta^0 \rightarrow \Delta^1$ induce inclusions $\Delta^0 \rightarrow \text{sd}^\bullet \Delta^1$ and evaluation maps $\text{ev}_i : A^{\text{sd}^\bullet \Delta^1} \rightarrow A^{\Delta^0} = A$. Let $f, g : A \rightarrow B$ be two homotopic $*$ -morphisms. As such, there exists a chain of $*$ -morphisms $f = f_0, f_1, \dots, f_n = g$ and homotopies $H_i : A \rightarrow B[t]$, $i = 0, \dots, n-1$ as in Definition 1.2.1. These homotopies can then be “concatenated” to an ind- $*$ -morphism $H : A \rightarrow B^{\text{sd}^\bullet \Delta^1}$. Conversely, it is easily seen that if two $*$ -morphisms $f, g : A \rightarrow B$ can be recovered from an ind- $*$ -morphism $H : A \rightarrow B^{\text{sd}^\bullet \Delta^1}$ by composition with the evaluation maps

$$\text{ev}_0 H = f \quad \text{ev}_1 H = g$$

then f and g are homotopic.

Definition 1.3.5. Let $A, B \in \text{ind} - \text{Alg}_\ell^*$, we write

$$[A, B] = \text{hom}_{\text{ind} - [\text{Alg}_\ell^*]}(A, B).$$

Lemma 1.3.6. Let X, Y be sets and $f, g : X \rightarrow Y$ bijections. Write, $f_*, g_* : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ as in Example 1.3.2. Then $[f_*] = [g_*] \in [\mathcal{M}_X, \mathcal{M}_Y]$.

Proof. Since homotopy is compatible with composition, we can reduce to the case when $X = Y$ and $g = \text{id}_X$. The matrix

$$u = \sum_{x \in X} e_{f(x), x}$$

is a unitary element of Γ_X and f_* is the restriction of $\text{ad}(u)$ (tensored with the identity). Then $i_+ \text{ad}(u) = \text{ad}(u \oplus 1) i_+$. Using [CT07, Section 3.4] there is a

homotopy $(v_0, v_1) \in M_2\Gamma_X[t]^2$ of multipliers between $\text{ad}(u \oplus 1)$ and $\text{ad}(1 \oplus u)$; thus, using Lemma 1.2.3, we have that $i_+^{0,2} \text{ad}(u \oplus 1)$ is $*$ -homotopic to $i_+^{0,2} \text{ad}(1 \oplus u)$. Hence

$$i_+^{0,2} i_+ = i_+^{0,2} \text{ad}(1 \oplus u) i_+ \sim i_+^{0,2} \text{ad}(u \oplus 1) i_+ = i_+^{0,2} i_+ \text{ad}(u)$$

and $\text{ad}(u)$ induces the identity in $[\mathcal{M}_X, \mathcal{M}_X]$. \square

1.4 Extensions

A $*$ -algebra can be regarded as a set or an ℓ -module in each case with or without involution. Each of these four choices gives rise to an underlying category \mathfrak{U} and a forgetful functor $F : \text{Alg}_\ell^* \rightarrow \mathfrak{U}$ which admits a left adjoint $\tilde{T} : \mathfrak{U} \rightarrow \text{Alg}_\ell^*$ that is the free $*$ -algebra functor for such F . We write $T = \tilde{T}F$. For the rest of this thesis we will fix one of the four choices as above for \mathfrak{U} , F and \tilde{T} .

An *extension* of $*$ -algebras is a sequence in Alg_ℓ^*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \quad (1.4.1)$$

where α is an isomorphism onto $\ker \beta$ and $C = \text{im } \beta$.

We say that a surjective $*$ -morphism is *split*, if it has a right inverse; we say that a surjective $*$ -morphism f is *semi-split* if $F(f)$ has right inverse in \mathfrak{U} . We say an extension (1.4.1) is semi-split if β is.

For ind- $*$ -algebras, a similar definition applies: a sequence in $\text{ind} - \text{Alg}_\ell^*$

$$0 \rightarrow (A_i) \xrightarrow{\alpha} (B_j) \xrightarrow{\beta} (C_k) \rightarrow 0 \quad (1.4.2)$$

is an extension of ind- $*$ -algebras if α a kernel for β and β is a cokernel for α . It is split if β admits a splitting and it is semi-split if $F(\beta)$ admits a splitting in $\text{ind} - \mathfrak{U}$.

Remark 1.4.3. If the underlying category \mathfrak{U} is the category of sets then every extension is semi-split, since every surjective map admits a section.

Remark 1.4.4. If ℓ satisfies the λ -assumption 1.1.24, then for a $*$ -morphism $f : A \rightarrow B$ for which $F(f)$ admits a splitting s , the splitting can be averaged as $s' = \lambda s + \lambda^* s^*$ in order to have an *involution preserving* splitting. Therefore, in this case, if f admits an ℓ -linear splitting, then it is semi-split for any choice of \mathfrak{U} and F .

Example 1.4.5. Let $A \in \text{Alg}_\ell^*$, we call the sequence

$$0 \rightarrow PA \rightarrow A[t] \xrightarrow{\text{ev}_0} A \rightarrow 0 \quad (1.4.6)$$

the *path extension*. It is split by the inclusion $A \subset A[t]$.

We call the sequence

$$0 \rightarrow \Omega A \rightarrow PA \xrightarrow{\text{ev}_1} A \rightarrow 0 \quad (1.4.7)$$

the *loop extension*. It admits an involution preserving ℓ -linear splitting $s(a) = ta$.

Let $f : A \rightarrow B$ be a $*$ -morphism. The *mapping path extension* of f is the extension induced by the pullback of the path extension of B along f

$$\begin{array}{ccccc} \Omega B & \longrightarrow & PB \times_B A & \xrightarrow{\pi_1} & A \\ \parallel & & \downarrow \pi_0 & \lrcorner & \downarrow f \\ \Omega B & \longrightarrow & PB & \xrightarrow{\text{ev}_1} & B. \end{array} \quad (1.4.8)$$

We call $P_f := PB \times_B A$ the *path algebra* of f . The mapping path extension has a natural ℓ -linear involution preserving splitting $s(a) = (tf(a), a)$. There is also natural inclusion $i_f : \ker(f) \rightarrow P_f$ given by $i_f(x) = (0, x)$. The same applies to the subdivided version which we write as $\mathcal{P}_f := \mathcal{P}B \times_B A$.

Example 1.4.9. Let X be a set and $A \in \text{Alg}_\ell^*$. We call the sequence

$$0 \rightarrow M_X A \rightarrow \Gamma_X A \rightarrow \Sigma_X A \rightarrow 0$$

the *cone extension*. By [CT07, first paragraph of p.92] it admits an ℓ -linear splitting.

Let $f : A \rightarrow B$ be a $*$ -morphism. The *cone map extension* of f is the extension induced by the pullback of the cone extension of B along $\Sigma_X f$.

$$\begin{array}{ccccc} M_X B & \longrightarrow & \Gamma_X B \times_B \Sigma A & \xrightarrow{\pi_1} & \Sigma_X A \\ \parallel & & \downarrow \pi_0 & \lrcorner & \downarrow \Sigma_X f \\ M_X B & \longrightarrow & \Gamma_X B & \longrightarrow & \Sigma_X B. \end{array} \quad (1.4.10)$$

We call $\Gamma_{X,f} := \Gamma_X B \times_B \Sigma A$ the *cone algebra* of f . The cone map extension has an ℓ -linear splitting given by composing $\Sigma_X f \times \text{id} : \Sigma_X A \rightarrow \Sigma_X B \times \Sigma_X A$ and the splitting $\Sigma_X B \rightarrow \Gamma_X B$. As before, when $X = \mathbb{N}$ we omit it from notation.

For every algebra morphism $f : A \rightarrow B$, the underlying map in \mathfrak{U} , $F(f) : F(A) \rightarrow F(B)$ induces a map $\tilde{f} : TA \rightarrow B$. In particular, for $\text{id} : A \rightarrow A$, we have a natural surjective transformation $\eta_A : T(A) \rightarrow A$. Set

$$J(A) := \ker(\eta_A),$$

this defines a functor $J : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell^*$. The *universal extension* of A is the extension

$$0 \rightarrow J(A) \rightarrow T(A) \xrightarrow{\eta_A} A \rightarrow 0$$

which is semi-split by the natural inclusion $s : A \rightarrow T(A)$.

For a semi-split extension

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

and a splitting s of $F(g)$, define $\hat{\xi} := \eta_B T'(s) : T(C) \rightarrow B$. The restriction of $\hat{\xi}$ to $J(C)$ maps to A since

$$g\hat{\xi} = g\eta_B T'(s) = \eta_C T(g)T'(s) = \eta_C.$$

Write ξ for the restriction of $\widehat{\xi}$ to $J(C)$. We call this map the *classifying map of the extension*. There is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \xi \uparrow & & \widehat{\xi} \uparrow & & \parallel \\ 0 & \longrightarrow & J(C) & \longrightarrow & T(C) & \xrightarrow{\eta_C} & C \longrightarrow 0. \end{array}$$

The definition of the classifying map ξ is clearly dependent of the splitting map s ; however, its homotopy class does not depend on s . Let s_1 and s_2 two different splittings of g and ξ_1 and ξ_2 be the corresponding classifying maps. Define $H : F(C) \rightarrow F(A[t])$ as

$$H(c) = (1 - t)\widehat{\xi}_1(c) + t\widehat{\xi}_2(c).$$

Extend H to a $*$ -homomorphism $H : T(C) \rightarrow A[t]$ by adjunction. This map is an elementary $*$ -homotopy between $\widehat{\xi}_1$ and $\widehat{\xi}_2$ and therefore ξ_1 and ξ_2 are homotopic; thus, the classifying map is natural up to homotopy. This shows the reasoning in calling the universal extension and the classifying map as such.

For an extension of ind- $*$ -algebras, the same reasoning applies and thus for any extension of ind- $*$ -algebras, there is also a unique classifying map in $\text{ind} - [Alg_\ell^*]$.

Remark 1.4.11. Take the following commutative diagram in Alg_ℓ^*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \longrightarrow 0 \end{array}$$

where each row is a semi-split extension. Let ξ be the classifying map associated to the first row extension and ξ' the classifying map associated the second row extension. Due to the uniqueness of the classifying map, the square

$$\begin{array}{ccc} J(C) & \xrightarrow{\xi} & A \\ \downarrow J(\gamma) & & \downarrow \alpha \\ J(C') & \xrightarrow{\xi'} & A' \end{array}$$

is commutative up to homotopy.

Example 1.4.12. Let $A, B \in Alg_\ell^*$ such that B is flat as an ℓ -module. Then, the extension

$$0 \rightarrow J(A) \otimes B \rightarrow T(A) \otimes B \rightarrow A \otimes B \rightarrow 0$$

is semi-split and we write the classifying map as

$$\phi_{A,B} : J(A \otimes B) \rightarrow J(A) \otimes B.$$

In the case the underlying category \mathfrak{U} is the category of ℓ -modules, this map is natural in A and B (up to homotopy).

Taking $B = \ell^X$ for some simplicial set X , we obtain a map $J(A^X) \rightarrow J(A)^X$. Similarly, for a pointed simplicial set (X, x) we obtain a map $J(A^{(X,x)}) \rightarrow J(A)^{(X,x)}$.

The loop extension (1.4.7) is a particular case of this setting, taking into account the identifications at the end of Example 1.1.22. We write the classifying map of the loop extension (1.4.7) as

$$\rho_A : J(A) \rightarrow \Omega A. \quad (1.4.13)$$

This map also induces an ind- $*$ -algebra map by composing ρ_A with the last vertex map $h_* : \Omega A \rightarrow A^{\mathbb{S}^1}$. As an abuse of notation we will write it as $\rho_A : J(A) \rightarrow A^{\mathbb{S}^1}$.

For a map $f : A \rightarrow B$, the classifying map of the mapping path extension (1.4.8) is $\rho_f := \rho_B \circ J(f)$; this can be seen using Remark 1.4.11. The same applies for the subdivided version.

Example 1.4.14. For each A the sequence

$$0 \rightarrow J(A)^{\mathbb{S}^1} \rightarrow T(A)^{\mathbb{S}^1} \rightarrow A^{\mathbb{S}^1} \rightarrow 0$$

is a semi-split extension as in Example 1.4.12. We write

$$\gamma_A : J(A)^{\mathbb{S}^1} \rightarrow J(A)^{\mathbb{S}^1} \quad (1.4.15)$$

for the classifying map of said extension. For $m, n \geq 0$, write

$$\gamma_A^{1,n} : J(A^{\mathbb{S}^n}) \rightarrow J(A)^{\mathbb{S}^n}$$

for the composition

$$J(A^{\mathbb{S}^n}) \xrightarrow{\gamma_A^{\mathbb{S}^n}} J(A^{\mathbb{S}^{n-1}})^{\mathbb{S}^1} \xrightarrow{\gamma_A^{\mathbb{S}^{n-1} \otimes \mathbb{S}^1}} J(A^{\mathbb{S}^{n-2}})^{\mathbb{S}^2} \rightarrow \dots \rightarrow J(A^{\mathbb{S}^1})^{\mathbb{S}^{n-1}} \xrightarrow{\gamma_A^{\otimes \mathbb{S}^{n-1}}} J(A)^{\mathbb{S}^n},$$

and $\gamma_A^{m,n} : J^m(A^{\mathbb{S}^n}) \rightarrow J^m(A)^{\mathbb{S}^n}$ for the composition

$$J^m(A^{\mathbb{S}^n}) \xrightarrow{J^{m-1}(\gamma_A^{1,n})} J^{m-1}(J(A)^{\mathbb{S}^n}) \xrightarrow{J^{m-2}(\gamma_{J(A)}^{1,n})} J^{m-2}(J^2(A)^{\mathbb{S}^n}) \rightarrow \dots \rightarrow J(J^{m-1}(A)^{\mathbb{S}^n}) \xrightarrow{\gamma_{J^{m-1}(A)}^{1,n}} J^m(A)^{\mathbb{S}^n}$$

1.5 *-Quasi-homomorphisms

Definition 1.5.1. Let $A, B \in \text{Alg}_\ell^*$, $C \trianglelefteq B$ a $*$ -ideal and $f_+, f_- : A \rightarrow B$ two $*$ -morphisms. We say that the pair $(f_+, f_-) : A \rightrightarrows B \supseteq C$ is a **-quasi-homomorphism* if $f_+(a) - f_-(a) \in C$ for every $a \in A$. This is equivalent to the following statement: if $\pi : B \rightarrow B/C$ is the quotient map, then $\pi f_+ = \pi f_-$.

Example 1.5.2. Recall from Example 1.1.5 the algebras $Q(A)$ and $q(A)$. By definition, there is a $*$ -quasi-homomorphism induced by the inclusions $\iota_0, \iota_1 : A \rightarrow Q(A)$:

$$(\iota_0, \iota_1) : A \rightrightarrows Q(A) \supseteq q(A).$$

This $*$ -quasi-homomorphism is universal in the following sense: let $(f_+, f_-) : A \rightrightarrows B \supseteq C$ be a $*$ -quasi-homomorphism. Then there is a natural map $f_+ \amalg f_- : Q(A) \rightarrow B$. Since $f_+ \amalg f_-$ maps $q(A)$ into C , we can compose to get

$$\begin{aligned} f_+ \amalg f_- \circ \iota_0 &= f_+ \text{ and} \\ f_+ \amalg f_- \circ \iota_1 &= f_-. \end{aligned}$$

We call the restriction of $f_+ \amalg f_-$ to $f : q(A) \rightarrow C$ the *classifying map* of the $*$ -quasi-homomorphism (f_+, f_-) .

Let \mathfrak{C} be an abelian category. A functor $H : Alg_\ell^* \rightarrow \mathfrak{C}$ is *split-exact* if for every split-exact extension

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

the sequence

$$0 \rightarrow H(A) \rightarrow H(B) \rightarrow H(C) \rightarrow 0$$

is exact in \mathfrak{C} .

Proposition 1.5.3 ([CMR07, Section 3.1.1]). *Let \mathfrak{C} be an abelian category and $E : Alg_\ell^* \rightarrow \mathfrak{C}$ a split-exact functor.*

- For every $*$ -quasi-homomorphism $(f_+, f_-) : A \rightrightarrows B \supseteq C$ there exists a morphism

$$E(f_+, f_-) : E(A) \rightarrow E(C)$$

induced by $E(f_+) - E(f_-) : E(A) \rightarrow E(B)$.

- $E(f_+, 0) = E(f_+)$.
- If $f_+ = f_- + g$ where $g(a)f_-(a) = f_-(a)g(a) = 0$ for every $a \in A$ then $E(f_+, f_-) = E(g)$.
- If $f : q(A) \rightarrow C$ is the classifying map of (f_+, f_-) then

$$E(f_+, f_-) = E(f) \circ E(\iota_0, \iota_1).$$

1.6 Stability

Definition 1.6.1. Let $F_1, F_2 : Alg_\ell^* \rightarrow Alg_\ell^*$, $G : Alg_\ell^* \rightarrow \mathfrak{C}$ be functors, $i : F_1 \rightarrow F_2$ be a natural transformation and $A \in Alg_\ell^*$. We say that the functor G is *i -stable at A* if the map $G(i_A) : G(F_1(A)) \rightarrow G(F_2(A))$ is an isomorphism. We say that G is *i -stable* if it is i -stable at every $A \in Alg_\ell^*$.

Example 1.6.2. A functor F is homotopy invariant as in Definition 1.2.2 if and only if it is stable for the canonical inclusion $A \rightarrow A[t]$.

Example 1.6.3. Let X be a set, $x, y \in X$ and $e_{x,y} \in M_X$ the matrix unit

$$e_{x,y}(z, w) = \delta_{(x,y),(z,w)}.$$

There is a natural map $i_x : \text{id}_{\text{Alg}_\ell^*} \rightarrow M_X$ defined as

$$\begin{aligned} i_{x,A} : A &\rightarrow M_X A \\ a &\mapsto e_{x,x} \otimes a. \end{aligned}$$

Lemma 1.6.4. *Let X be a set and i_x be as in Example 1.6.3. If a functor $G : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ is i_x -stable for some x then it is i_y stable for any $y \in X$. Moreover $G(i_x) = G(i_y)$ for any $x, y \in X$.*

Proof. We follow [Cor11, Lemma 2.2.4]. There are permutation matrices $\sigma_2, \sigma_3 \in M_X A \otimes M_X A$ of orders two and three such that both conjugate $(i_{x, M_X A} \otimes \text{id}_{M_X A})i_{x,A}$ into $(i_{x, M_X A} \otimes \text{id}_{M_X A})i_{y,A}$. Since permutation matrices are unitary, conjugation by σ_2 and σ_3 are $*$ -isomorphisms. After applying G we get

$$\begin{aligned} G(\text{ad}(\sigma_2))G((i_{x, M_X A} \otimes \text{id}_{M_X A})i_{x,A}) &= G((i_{x, M_X A} \otimes \text{id}_{M_X A})i_{y,A}) \\ &= G(\text{ad}(\sigma_3))G((i_{x, M_X A} \otimes \text{id}_{M_X A})i_{x,A}) \end{aligned} \quad (1.6.5)$$

Since the orders of σ_2 and σ_3 are coprime and all the maps in (1.6.5) are isomorphisms, it follows that $G(\text{ad}(\sigma_2))$ and $G(\text{ad}(\sigma_3))$ are equal to the identity. Furthermore, since $G(i_{x, M_X A} \otimes \text{id}_{M_X A})$ is an isomorphism, we get that $G(i_{x,A}) = G(i_{y,A})$. \square

Definition 1.6.6. We say that G is M_X -stable if it is i_x -stable for some (therefore, for any) $x \in X$. In this case we write i_X for any i_x . If the set X is fixed, we simply write i . When X has cardinality n we write $i_n : \text{id} \rightarrow M_n$ for i_X .

Lemma 1.6.7. *Let X be a set and $x, y \in X$. Then the maps $i_+ i_x, i_+ i_y : \ell \rightarrow M_\pm M_X$ are $*$ -homotopic.*

Proof. Assume $x \neq y$ and let $X' = X \setminus \{x, y\}$. Let

$$u = e_{y,x} - e_{x,y} + \sum_{z \in X'} e_{z,z}.$$

It is easily seen that u is unitary in Γ_X and satisfies $\text{ad}(u)i_x = i_y$. Moreover, there is a rotational homotopy $u(t) \in \Gamma_X[t]$ [CT07, Section 3.4] such that $u(0) = 1$ and $u(1) = u$. Then, using Lemma 1.2.3 we obtain the desired statement. \square

Lemma 1.6.8. *Let X be a set with at least two elements, $H : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ be an M_X -stable functor, $A \subseteq B \in \text{Alg}_\ell^*$ and $u \in B$ such that*

$$\begin{aligned} uA, Au^* &\subseteq A \text{ and} \\ au^*u' &= aa' \text{ for any } a, a' \in A. \end{aligned}$$

Then $\text{ad}(u) : A \rightarrow A$ is a $$ -homomorphism and $H(\text{ad}(u)) = \text{id}_{H(A)}$.*

Proof. The argument is as in [Cor11, Proposition 2.2.6]. We can assume B is unital (changing B for \tilde{B}). Consider $u \oplus 1 \in M_2 B$ and observe that $\text{ad}(u \oplus 1) : M_2 A \rightarrow M_2 A$ is a $*$ -homomorphism. Also, if $i_0 : A \rightarrow M_2 A$ and $i_1 : A \rightarrow M_2 A$ are the inclusions in the upper left corner and lower right corner respectively then

$$\begin{aligned} \text{ad}(u \oplus 1)i_0 &= i_0 \text{ad}(u) \text{ and} \\ \text{ad}(u \oplus 1)i_1 &= i_1. \end{aligned}$$

Due to Lemma 1.6.4, applying G we get that $G(i_0) = G(i_1)$ are isomorphisms. Therefore,

$$\begin{aligned} G(i_0)G(\text{ad}(u)) &= G(\text{ad}(u \oplus 1))G(i_0) \\ &= G(\text{ad}(u \oplus 1))G(i_1) \\ &= G(i_1) \\ &= G(i_0); \end{aligned}$$

so $G(\text{ad}(u))$ is the identity. \square

Lemma 1.6.9. *Let X be a set with at least two elements. Let \mathfrak{C} be category enriched over abelian groups and $H : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ be an M_X -stable functor. Then the map for two distinct $x, y \in X$, the map*

$$H(A \oplus A) \xrightarrow{H(i_x \oplus i_y)} H(M_X A) \xrightarrow{H(i_x)^{-1} = H(i_y)^{-1}} H(A)$$

induces the additive operation on $H(A)$,

Proof. Write $D = i_x \oplus i_y : A \oplus A \rightarrow M_X A$ and $\nabla : H(A) \oplus H(A) \rightarrow H(A)$ for the operation in \mathfrak{C} . Using Lemma 1.6.4, the diagram

$$\begin{array}{ccc} H(A) \oplus H(A) & \xrightarrow{\text{inc}_1 \oplus \text{inc}_2} & H(A \oplus A) \\ \downarrow \nabla & & \downarrow H(D) \\ H(A) & \xrightarrow{H(i_x) = H(i_y)} & H(M_X A) \end{array}$$

commutes \square

Lemma 1.6.10. *Let X, Y be two sets such that X has at least two elements and Y has greater cardinality than X . Then, any M_Y -stable functor $G : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ is also M_X -stable.*

Proof. Since a bijection between sets induces a $*$ -isomorphism between their matrix algebras, we might assume that $X \subseteq Y$. We will prove the lemma in the case the coefficients are $A = \ell$, the same proof applies for any coefficients. Write $\text{inc} : X \hookrightarrow Y$ for the natural inclusion map. Let $x \in X$ and $i = i_x : \ell \rightarrow M_X$. Since G is M_Y -stable, $G(\text{inc} \circ i)$ is an isomorphism and therefore, $G(i)$ is a split monomorphism and $G(\text{inc})$ is a split epimorphism.

Let $\tau : M_X \otimes M_Y \rightarrow M_Y \rightarrow M_X$ defined by $\tau(a \otimes b) = b \otimes a$. We have

$$\tau(i \otimes \text{id}_{M_Y})\text{inc} = \text{inc} \otimes i. \quad (1.6.11)$$

Let $\sigma : Y \times X \rightarrow Y \times X$ be any bijection that restricts to coordinate permutation on $X \times \{x\}$. Also write σ for the corresponding permutation matrix in $M_{Y \times X} = M_Y \otimes M_X$. Then we have

$$\text{ad}(\sigma)(\text{inc} \otimes i) = i \otimes \text{id}_{M_X}.$$

Since $G(\text{ad}(\sigma))$ is the identity due to Lemma 1.6.4 and $G(i \otimes \text{id}_{M_X})$ is an isomorphism, it follows that $G(\text{inc} \otimes i)$ is an isomorphism. Using (1.6.11) we get that $G(\text{inc})$ is also an split monomorphism, and therefore an isomorphism. Since $G(\text{inc} \circ i)$ is an isomorphism it follows that $G(i)$ is an isomorphism and that concludes the proof. \square

Definition 1.6.12. Let $A \in \text{Alg}_\ell^*$ and $G : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ be a functor. We say that G is *hermitian stable on A* if for every embedding $A \trianglelefteq R$ as a $*$ -ideal in a unital $*$ -algebra R , every central unitary element $\varepsilon \in R$ and any two invertible ε -hermitian elements $\phi, \psi \in R$, the functor G maps the upper left corner inclusion

$$i_\phi : A^\phi \rightarrow (M_2A)^{(\phi \oplus \psi)}$$

to an isomorphism.

Remark 1.6.13. Taking $\varepsilon = 1$, $R = \tilde{A}$ and $\phi = \psi = 1$ in the previous definition, we get that any hermitian stable functor is also $i_2 : \text{id} \rightarrow M_2$ stable

Remark 1.6.14. Let (P, ψ) and (Q, χ) be hermitian modules as in Example 1.1.17. Using (1.1.18), it follows that a hermitian stable functor G sends the map induced by the inclusion

$$\text{End}(P) \otimes A \rightarrow \text{End}(P \oplus Q) \otimes A$$

to an isomorphism.

Proposition 1.6.15 ([cf. Ell14, Proposition 3.1.9]). *Suppose ℓ satisfies the λ -assumption 1.1.24 and let $G : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ be a M_2 -stable functor. Then $G \circ M_\pm$ is hermitian stable.*

Proof. Since ℓ satisfies the λ -assumption, we can use Remark 1.1.26 to get isomorphisms

$$\begin{aligned} M_\pm A^\phi &\cong {}_\varepsilon M_2 A \text{ and} \\ M_\pm (M_2 A)^{(\phi \oplus \psi)} &\cong {}_\varepsilon M_2 M_2 A. \end{aligned}$$

Using the commutative diagram

$$\begin{array}{ccccc} M_\pm A^\phi & \xrightarrow{\sim} & {}_\varepsilon M_2 A & & \\ \text{id}_{M_\pm} \otimes i_\phi \downarrow & & \text{id}_{{}_\varepsilon M_2} \otimes i_2 \downarrow & \searrow i_2 & \\ M_\pm (M_2 A)^{(\phi \oplus \psi)} & \xrightarrow{\sim} & {}_\varepsilon M_2 M_2 A & \xrightarrow{\sim} & M_2({}_\varepsilon M_2 A) \end{array}$$

and the fact that i_2 is mapped to an isomorphism through G , we get that $G \circ M_\pm(i_\phi)$ is an isomorphism as desired. \square

Corollary 1.6.16. *Assuming ℓ and $G : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ as in Proposition 1.6.15, if G is also i_+ -stable, then G is hermitian stable.*

Proof. Since we have the commutative diagram

$$\begin{array}{ccc} A^\phi & \xrightarrow{i_+} & M_\pm A^\phi \\ \downarrow i_\phi & & \downarrow \text{id}_{M_\pm} \otimes i_\phi \\ (M_2 A)^{\phi \oplus \psi} & \xrightarrow{i_+} & M_\pm (M_2 A)^{\phi \oplus \psi}, \end{array}$$

using that $G(i_+)$ is an isomorphism and by Proposition 1.6.15 we also have $G(\text{id}_{M_\pm} \otimes i_\phi)$ is an isomorphism we get that $G(i_\phi)$ is an isomorphism. \square

Chapter 2

Hermitian Algebraic K -theory

In this chapter we recall the definition of the hermitian algebraic K -theory spectra K^h together with some properties. We also recall the definition of the Karoubi-Villamayor hermitian K -theory KV^h and we construct the analogue to Weibel's homotopy K -theory for the hermitian case KH^h . In Section 2.2 we also recall the product structure of K^h and how it passes to KH^h . Finally in Section 2.3 we recall Karoubi's Fundamental Theorem with some associated reformulations and how it passes to KH^h ; we will use this later in Chapter 5.

2.1 Definitions

Let A a $*$ -ring. We write

$$\mathcal{U}(A) = \{x \in A : x^*x = xx^*, x + x^* + xx^* = 0\}.$$

The set $\mathcal{U}(A)$ is a group under the operation

$$x \cdot y = x + y + xy.$$

When A is unital, the group $\mathcal{U}(A)$ is isomorphic to the group of unitary elements of A via the map $x \rightarrow 1 + x$.

Let R be a unital ring, $A \trianglelefteq R$ a $*$ -ideal and $\varepsilon \in R$ central unitary. Put

$${}_{\varepsilon}\mathcal{O}(A) = \mathcal{U}({}_{\varepsilon}M_2M_{\infty}A).$$

By (1.1.20) we have a group isomorphism

$${}_{\varepsilon}\mathcal{O}(A) \cong {}_1\mathcal{O}({}_{\varepsilon}M_2A). \quad (2.1.1)$$

The ε -hermitian K -theory groups of a unital $*$ -ring R are the stable homotopy groups of a spectrum ${}_{\varepsilon}K^hR = \{{}_{\varepsilon}K^hR_n\}$ whose n -th space is ${}_{\varepsilon}K^hR_n = \Omega B_{\varepsilon}\mathcal{O}(\Sigma^{n+1}R)^+$, the loop space of the $+$ -construction [see Lod76, Section 3.1.6]. As usual we also write

$${}_{\varepsilon}K_n^h(R) = \pi_n({}_{\varepsilon}K^hR) \quad (n \in \mathbb{Z})$$

for the n -th stable homotopy group. When $\varepsilon = 1$ we drop it from the notation. For a nonunital $*$ -ring A , we put

$$\pm_1 K_n^h(A) = \ker(\pm_1 K_n^h(\tilde{A}_{\mathbb{Z}}) \rightarrow \pm K_n^h(\mathbb{Z})). \quad (2.1.2)$$

If A is unital, these groups agree with those defined above since in that case $\tilde{A}_{\mathbb{Z}} \cong A \times \mathbb{Z}$ and using the fact that $+$ -construction is additive, the kernel in (2.1.2) recovers $\pm_1 K_n^h(A)$.

A ring A is called K -excisive if for any embedding $A \trianglelefteq R$ as an ideal of a unital ring R and every unital homomorphism $R \rightarrow S$ mapping A isomorphically onto an ideal of S , the map of relative K -theory spectra $K(R : A) \rightarrow K(S : A)$ is an equivalence. The definition of a $K^h = {}_1 K^h$ -excisive $*$ -ring is analogous.

Remark 2.1.3. Let A be a K -excisive ring that is a $*$ -algebra over ℓ , and suppose that ℓ satisfies the λ -assumption 1.1.24. Let $A \trianglelefteq R$ be a $*$ -ideal embedding into a unital $*$ -algebra and $f : R \rightarrow S$ be a unital $*$ -algebra homomorphism mapping A isomorphically onto a $*$ -ideal of S and $\varepsilon \in \ell$ be a central unitary. By [Bat11, Corollary 3.5.1] the map ${}_{\varepsilon} K^h(R : A) \rightarrow {}_{\varepsilon} K^h(S : A)$ is an equivalence. In particular, if A is K -excisive then it is also K^h -excisive. Taking all this into account, and assuming that ℓ satisfies the λ -assumption 1.1.24, we set, for any K -excisive $A \in \text{Alg}_{\ell}^*$, unitary $\varepsilon \in \ell$ and $n \in \mathbb{Z}$,

$${}_{\varepsilon} K_n^h(A) = \ker({}_{\varepsilon} K_n^h(\tilde{A}) \rightarrow {}_{\varepsilon} K_n^h(\ell)). \quad (2.1.4)$$

Remark 2.1.5. For $n \leq 0$ and not necessarily K -excisive A , we take (2.1.4) as a definition. The non-positive hermitian K -groups agree with Bass' quadratic K -groups [Bas73] for the maximum form parameter. In particular, by [Bas73, Chapter III, Theorem 1.1] hermitian K -theory as defined above satisfies excision in non-positive dimensions.

Remark 2.1.6. Let R be a unital $*$ -ring. Suppose that R has an element λ that satisfies the λ -assumption 1.1.24. Let $S \in \Sigma$ be the class of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using the fact that the cup product with $[S] \in K_1^h(\Sigma)$ induces an isomorphism $K_0^h(R) \cong K_1^h(\Sigma R)$ [Lod76, Théorème 3.1.7], the group $K_0^h(R)$ can be described as the set of formal differences $[p] - [q]$ where $p, q \in {}_1 M_2 M_{\infty} R$ are projections and $[p] = [p']$ if there is a unitary matrix $u \in {}_1 M_2 M_n R$ such that u conjugates p into p' [KV73, Section 2].

For a class $x = [p] - [q] \in K_0^h(R)$ there are $*$ -morphisms $p, q : \mathbb{Z} \rightarrow {}_1 M_2 M_{\infty} R$ mapping 1 to p and q respectively. These $*$ -morphisms induce maps $p_*, q_* : K_0^h(\mathbb{Z}) \rightarrow K_0^h(R)$ sending the class of $[1]$ to $[p]$ and $[q]$ respectively. This implies that the $*$ -quasi-homomorphism $(p, q) : \mathbb{Z} \rightrightarrows {}_1 M_2 M_{\infty} R \supseteq 0$, has an associated map $(p_*, q_*) =$

$p_* - q_* : K_0^h(\mathbb{Z}) \rightarrow K_0^h(R)$ which maps the class of [1] to x . This then implies that the set of $*$ -quasi-homomorphisms $\{\mathbb{Z} \rightrightarrows {}_1M_2M_\infty R\}$ maps surjectively onto $K_0^h(R)$ sending each pair of $*$ -quasi-homomorphisms to their corresponding associated map evaluated at the class [1]. Since K_0^h satisfies excision, it follows that the same applies to any $*$ -ring A : the set

$$qq(\mathbb{Z}, A) := \{\mathbb{Z} \rightrightarrows {}_1M_2M_\infty \tilde{A}_\mathbb{Z} \supseteq {}_1M_2M_\infty A\}$$

maps surjectively to $K_0^h(A)$. If $A \in Alg_\ell^*$ then the same holds with ℓ substituted for \mathbb{Z} and ℓ -linear, $*$ -quasi-homomorphisms.

For a $*$ -ring A and $\varepsilon = \pm 1$, Karoubi and Villamayor also introduce hermitian K -groups for $n \geq 1$. They agree with the homotopy groups of the simplicial group ${}_\varepsilon\mathcal{O}(A^\Delta)$ up to a degree shift

$${}_\varepsilon KV_n^h(A) = \pi_{n-1\varepsilon}\mathcal{O}(A^\Delta) \quad (n \geq 1).$$

The argument of [Cor11, Proposition 10.2.1] shows that the definition above is equivalent to that given in [KV73]; we have

$${}_\varepsilon KV_{n+1}^h(A) = {}_\varepsilon KV_1^h(\Omega^n A) \quad (n \geq 1).$$

Similarly, if A is unital, for all $n \geq 1$ we have

$${}_\varepsilon KV_n^h(A) = \pi_n B_\varepsilon \mathcal{O}(A^\Delta) = \pi_n B_\varepsilon \mathcal{O}(A^\Delta)^+ = \pi_n \Omega B_\varepsilon \mathcal{O}(\Sigma A^\Delta)^+. \quad (2.1.7)$$

Applying ${}_\varepsilon K_n^h$ to the path extension (1.4.6) and using excision, we obtain a natural map

$${}_\varepsilon K_n^h(A) \rightarrow {}_\varepsilon K_{n-1}^h(\Omega A) \quad (n \leq 0).$$

For $n \in \mathbb{Z}$, the n^{th} homotopy ε -hermitian K -theory group of A is

$${}_\varepsilon KH_n^h(A) = \underset{m \geq n}{\text{colim}} {}_\varepsilon K_{-m}^h(\Omega^{m+n} A).$$

Remark 2.1.8. One can also describe ${}_\varepsilon KH_n^h$ in terms of ${}_\varepsilon KV^h$; by [KV73, Théorème 4.1], ${}_\varepsilon KV^h$ satisfies excision for the cone extension (1.4.10). Hence we have a map

$${}_\varepsilon KV_n^h(A) \rightarrow {}_\varepsilon KV_{n+1}^h(\Sigma A).$$

The argument of [CT07, Proposition 8.1.1] shows that

$${}_\varepsilon KH_n^h(A) = \underset{m}{\text{colim}} {}_\varepsilon KV_{n+m}^h(\Sigma^m A).$$

Now assume that A is unital; let ${}_\varepsilon KH(A)$ be the total spectrum of the simplicial spectrum ${}_\varepsilon K^h(A^\Delta)$. We have

$$\pi_n({}_\varepsilon KH^h(A)) = \underset{m}{\text{colim}} \pi_{n+m} \Omega B_\varepsilon \mathcal{O}(\Sigma^m A^\Delta)^+ = \underset{n}{\text{colim}} {}_\varepsilon KV_{n+m}^h(\Sigma^m A) = {}_\varepsilon KH_n^h(A).$$

For any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$$

there is a natural *index map* $\partial : K_1^h(B/A) \rightarrow K_0^h(A)$ [see Bas73, Chapter III].

Remark 2.1.9. There is a natural comparison map $c_m : K_m^h(A) \rightarrow KH_m^h(A)$. For $m \leq 0$ this is just mapping to the colimit. For $m > 0$ and A unital, using the description of (2.1.7) and the natural inclusion $A \rightarrow A^\Delta$, we get a comparison map $c'_m : K_m^h(A) \rightarrow KV_m^h$, then, by Remark 2.1.8, the comparison map factors

$$K_m^h(A) \xrightarrow{c'_m} KV_m^h(A) \rightarrow KH_m^h(A)$$

Using repeatedly the index map of the loop extension (1.4.7) we get maps

$$K_m^h(A) \rightarrow KV_m^h(A) \cong KV_1(\Omega^{m-1}A) \rightarrow KV_0^h(\Omega^m A) = K_0(\Omega^m A).$$

Finally, composing with the comparison map c_0 we arrive at $KH_0^h(\Omega^m A) \cong KH_m^h(A)$.

Lemma 2.1.10. *Homotopy hermitian K -theory is homotopy invariant, matricially stable and satisfies excision.*

Proof. The proof is the same as in non-hermitian K -theory [see Cor11, Theorem 5.1.1]. \square

Lemma 2.1.11. *Let $\varepsilon \in \ell$ be unitary. If either $n \leq 0$ or A is K^h -excisive, then there is a canonical isomorphism*

$${}_\varepsilon K_n^h(A) \cong K_n^h({}_\varepsilon M_2 A).$$

Moreover for all $A \in \text{Alg}_\ell^*$ we have a canonical isomorphism

$${}_\varepsilon KH_n^h(A) \cong KH_n^h({}_\varepsilon M_2 A) \quad (n \in \mathbb{Z}).$$

Proof. The isomorphism (2.1.1) is canonical up to the choices of an element $\lambda \in \ell$ in the λ -assumption 1.1.24 and a bijection $\{1, 2\} \times X \rightarrow X$. By [Lod76, Lemme 1.2.7], if A is unital, then varying those choices has no effect on the homotopy type of the induced isomorphism $B_\varepsilon \mathcal{O}(A)^+ \cong B_1 \mathcal{O}({}_\varepsilon M_2 A)^+$. Applying this to $\Sigma^r A$ we obtain the statement of the lemma for unital A . The nonunital case follows from the unital one using split-exactness. The statement for ${}_\varepsilon KH^h$ follows by applying the former case for $\Omega^r A$ and from the definition. \square

2.2 Cup products in KH^h

Hermitian K -theory of unital $*$ -rings is equipped with products [Lod76, Chapitre III]. Using that K^h satisfies excision in nonpositive dimensions we obtain, for $R, A \in \text{Alg}_\ell^*$ with R unital, $m \in \mathbb{Z}$ and $n \leq 0$, a natural product

$$K_m^h(R) \otimes_{\mathbb{Z}} K_n^h(A) \xrightarrow{*} K_{m+n}^h(R \otimes A). \quad (2.2.1)$$

If moreover $m \leq 0$, we also obtain the product above for not necessarily unital R .

Remark 2.2.2. Using Lemma 2.1.11, the product (2.2.1) also gives a product

$${}_{\varepsilon}K_m^h(R) \otimes_{\mathbb{Z}} {}_{\eta}K_n^h(A) \xrightarrow{\star} {}_{\varepsilon\eta}K_{m+n}^h(R \otimes A). \quad (2.2.3)$$

Remark 2.2.4. Let R, S be unital \ast -rings that satisfy λ -assumption 1.1.24, using the description of Remark 2.1.6, the cup product

$$K_0^h(R) \otimes_{\mathbb{Z}} K_0^h(S) \xrightarrow{\star} K_0^h(R \otimes S)$$

corresponds to the natural extension of scalars of projections [cf. Lod76, Section 3.1.4]: for $[p] \in \mathcal{V}_{\infty}^h R$ and $[q] \in \mathcal{V}_{\infty}^h S$

$$[p] \star [q] = [p \otimes q].$$

Lemma 2.2.5. *Let $R, S \in \text{Alg}_{\ell}^*$ be unital that satisfy the λ -assumption 1.1.24 and let $I \trianglelefteq S$ be a \ast -ideal. Assume that the sequence*

$$0 \rightarrow R \otimes I \rightarrow R \otimes S \rightarrow R \otimes (S/I) \rightarrow 0$$

is exact and let ∂ be the associated index map. Then the following diagram commutes

$$\begin{array}{ccc} K_0^h(R) \otimes_{\mathbb{Z}} K_1^h(S/I) & \xrightarrow{\star} & K_1^h(R \otimes (S/I)) \\ \downarrow \text{id} \otimes \partial & & \downarrow \partial \\ K_0^h(R) \otimes_{\mathbb{Z}} K_0^h(I) & \xrightarrow{\star} & K_0^h(R \otimes I). \end{array}$$

Proof. Because R is unital and satisfies the λ -assumption, we may regard $K_0^h(R)$ as the group completion of the monoid $\mathcal{V}_{\infty}^h(R)$ as in Remark 2.1.6. If $g \in {}_1M_2M_n(S/I)$ is unitary, $p \in {}_1M_2M_n R$ is a self-adjoint idempotent and $\mathbb{1}_n \in {}_1M_2M_n$ is the identity matrix, then (see [Wei13, Corollary 1.6.1] for the non-hermitian case)

$$[p] \star [g] = [p \otimes g + (\mathbb{1}_n - p) \otimes \mathbb{1}_n] \in K_1^h(R \otimes (S/I)) \quad (2.2.6)$$

On the other hand, for any lift $h \in \mathcal{U}({}_1M_2M_{2n}S)$ of $g \oplus g^{-1}$ we have

$$\partial[g] = [h\mathbb{1}_n h^{-1}] - [\mathbb{1}_n].$$

Choosing the lift for (2.2.6) as

$$p \otimes h + (\mathbb{1}_{2n} - (p \oplus p)) \otimes \mathbb{1}_{2n}$$

we obtain $\partial([p] \star [g]) = [p] \star \partial[g]$. \square

Lemma 2.2.7. *Suppose ℓ satisfies the λ -assumption 1.1.24. Let $m \in \mathbb{Z}$, $n \leq 0$ and $R, A \in \text{Alg}_{\ell}^*$ with R unital. Let ∂ be the connecting map associated to the path extension (1.4.6). Assume that $\max\{n, m+n\} \leq 0$. Then the following diagram commutes.*

$$\begin{array}{ccc} K_m^h(R) \otimes_{\mathbb{Z}} K_n^h(A) & \xrightarrow{\star} & K_{m+n}^h(R \otimes A) \\ \downarrow \text{id} \otimes \partial & & \downarrow \partial \\ K_m^h(R) \otimes_{\mathbb{Z}} K_{n-1}^h(\Omega A) & \xrightarrow{\star} & K_{m+n-1}^h(R \otimes \Omega A) \end{array}$$

Proof. Let $j_2 : \ell \rightarrow \ell \oplus \ell$ be the inclusion in the second summand. The path and loop extensions, (1.4.6) and (1.4.7) respectively, are connected by a map of extensions

$$\begin{array}{ccccc} \Omega & \longrightarrow & P & \xrightarrow{\text{ev}_1} & \ell \\ \parallel & & \downarrow \text{inc} & & \downarrow j_2 \\ \Omega & \longrightarrow & \ell[t] & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \ell \oplus \ell \end{array}$$

Let $i \leq 0$. Applying Lemma 2.2.5 with $S = \Sigma[t]$, $I = \Sigma\Omega$ and $R = \Sigma^{-i}\tilde{A}$, and using naturality and excision, we obtain that the boundary map $\partial : K_i^h(A) \rightarrow K_{i-1}^h(\Omega A)$ is the cup product with $\partial([1]) \in K_{-1}^h(\Omega)$. The proof now follows from associativity of \star . \square

Corollary 2.2.8. *Suppose ℓ satisfies the λ -assumption 1.1.24. Let $R, A \in \text{Alg}_\ell^*$ with R unital and let $m, n \in \mathbb{Z}$.*

i) *There is an associative product*

$$\star : K_m^h(R) \otimes_{\mathbb{Z}} KH_n^h(A) \rightarrow KH_{m+n}^h(R \otimes A).$$

ii) *Let $c_* : K_*^h(R) \rightarrow KH_*^h(R)$ be the comparison map. Then for all $m \in \mathbb{Z}$ and $\xi \in K_m^h(R)$, $c_m(\xi) = \xi \star c_0([1])$.*

Proof. Part i) is immediate from Lemma 2.2.7 upon taking colimits. For $m \leq 0$, part ii) is clear from the construction of \star and the definition of KH^h . For $m > 0$, this follows from Remark 2.1.9 and the fact that since $KV_{-1}^h = K_{-1}^h$, the diagram

$$\begin{array}{ccc} K_*^h(R) \otimes_{\mathbb{Z}} K_{-1}^h(\Omega) & \xrightarrow{\star} & K_{*-1}^h(\Omega R) \\ \downarrow c_* \otimes \text{id} & & \downarrow c_* \\ KV_*^h(R) \otimes_{\mathbb{Z}} KV_{-1}^h(\Omega) & \xrightarrow{\star} & KV_{*-1}^h(\Omega R) \end{array}$$

commutes. \square

Lemma 2.2.9. *Suppose ℓ satisfies the λ -assumption 1.1.24. Let $A, B \in \text{Alg}_\ell^*$ and $m, n \in \mathbb{Z}$. Then (2.2.1) induces an associative product*

$$KH_m^h(A) \otimes_{\mathbb{Z}} KH_n^h(B) \xrightarrow{\star} KH_{m+n}^h(A \otimes B).$$

If $m \leq 0$ or A is unital, then the following diagram commutes

$$\begin{array}{ccc} KH_m^h(A) \otimes_{\mathbb{Z}} KH_n^h(B) & \xrightarrow{\star} & KH_{m+n}^h(B) \\ \downarrow c_m \otimes 1 & \nearrow \star & \\ KH_m^h(A) \otimes_{\mathbb{Z}} KH_n^h(B) & & \end{array}$$

Proof. Lemma 2.2.7 shows that the boundary map $\partial : K_*^h \rightarrow K_{*+1}^h \circ \Omega$ is the cup product with $\partial([1]) \in K_{-1}^h(\Omega)$. It follows that for all $r \leq 0$, the following diagram commutes:

$$\begin{array}{ccc} K_r^h(A) \otimes_{\mathbb{Z}} K_r^h(B) & \xrightarrow{\quad * \quad} & K_{2r}^h(A \otimes B) \\ \downarrow \partial \otimes \partial & & \downarrow \partial^2 \\ K_{r-1}^h(\Omega A) \otimes_{\mathbb{Z}} K_{r-1}^h(\Omega B) & \xrightarrow{\quad * \quad} & K_{2r-2}^h(\Omega^2 A \otimes B). \end{array}$$

Taking colimit along the columns we get the desired product map for $r = s = 0$. The general case is obtained from the latter applying the suspension and loop functors as many times as appropriate. Commutativity of the diagram in the statement follows from Corollary 2.2.8. \square

Corollary 2.2.10. *Let $A \in \text{Alg}_\ell^*$ and $n \in \mathbb{Z}$, then ${}_\varepsilon K H_n^h(A)$ is a $K H_0^h(\ell)$ -module with the action induced by the product in Lemma 2.2.9.*

2.3 Karoubi's Fundamental Theorem

Let $A \in \text{Rings}^*$ and consider $\widehat{A} = \text{ind}(\text{res}(A)) = A \oplus A^{op}$ as in Example 1.1.6. There are natural $*$ -morphisms

$$\begin{aligned} \phi_A : \widehat{A} &\rightarrow M_2(A) \\ (a, b) &\mapsto \begin{pmatrix} a & 0 \\ 0 & b^* \end{pmatrix}, \end{aligned} \tag{2.3.1}$$

$$\begin{aligned} \eta_A : A &\rightarrow \widehat{A} \\ a &\mapsto (a, a^*). \end{aligned} \tag{2.3.2}$$

Write $U'A = \Gamma_{\phi_A}$ and $V'A = \Gamma_{\eta_A}$ as in Example 1.4.9. This defines functors $U', V' : \text{Rings}^* \rightarrow \text{Rings}^*$ and write $(U')^n, (V')^n$ ($n \geq 0$) for their repeated composition. As in Example 1.4.9, there are natural maps $U'A \rightarrow \Sigma \widehat{A}$ and $V'A \rightarrow \Sigma A$. The projection on the first coordinate $\widehat{A} \rightarrow A$ is *not* a $*$ -morphism but is a ring morphism and as such it induces a map $K(\widehat{A}) \rightarrow K(A)$. Since for a unital ring (not necessarily with involution) $\mathcal{U}(\text{inv}(R)) = \text{GL}(R)$, we have that ${}_\varepsilon K^h(\text{inv}(R)) \sim K(R)$ and therefore ${}_\varepsilon K^h(\widehat{R}) \sim K(R)$. It follows using the cone extension from Example 1.4.9 that there are maps

$$\begin{aligned} \Omega_\varepsilon K^h(U'R) &\rightarrow K(R) \xrightarrow{(\phi_R)_*} {}_\varepsilon K^h(R) \\ \Omega_\varepsilon K^h(V'R) &\rightarrow {}_\varepsilon K^h(R) \xrightarrow{(\eta_R)_*} K(R) \end{aligned}$$

and that $\Omega_\varepsilon K^h(U'R)$ and $\Omega_\varepsilon K^h(V'R)$ are the homotopic fibers of the maps $(\phi_R)_*$ and $(\eta_R)_*$ respectively.

Theorem 2.3.3 (Karoubi, [Kar80]). *There is an element $\theta_0 \in {}_{-1}K_2^h((U')^2\mathbb{Z})$ such that:*

i) *The composite*

$${}_{-1}K_2^h((U')^2\mathbb{Z}) \rightarrow {}_{-1}K_2^h(\widehat{\Sigma U'\mathbb{Z}}) \cong {}_{-1}K_1^h(\widehat{U'\mathbb{Z}}) \cong K_1(U'\mathbb{Z}) \rightarrow K_1(\widehat{\Sigma\mathbb{Z}}) \cong K_0(\widehat{\mathbb{Z}}) \xrightarrow{pr_1} K_0(\mathbb{Z}) = \mathbb{Z}$$

maps θ_0 to 1.

ii) *Assume that ℓ -satisfies the λ -assumption 1.1.24. Then, for every unital $*\text{-}\ell$ -algebra R , the product with θ_0 induces an isomorphism*

$$\theta_0 \star - : {}_{\varepsilon}K_*^h(R) \cong {}_{-\varepsilon}K_{*+2}^h((U')^2R).$$

Proof. The element θ_0 of the present theorem appears under the name of σ in the first line of [Kar80, Section 3.1]. Using the identifications

$$\Omega_{\varepsilon}K^h(U'V'R) \sim \Omega_{\varepsilon}K^h(V'U'R) \sim {}_{\varepsilon}K^h(R) \quad (2.3.4)$$

as mentioned by Karoubi in [Kar80, Section 1.4], the current theorem is just another way of phrasing Karoubi's fundamental theorem

$${}_{\varepsilon}K^h(V'R) \sim \Omega_{-\varepsilon}K^h(U'R).$$

Furthermore, the theorem as stated here is equivalent to that proved in [Kar80, Section 3.5], which says that product with θ_0 induces an isomorphism

$${}_{\varepsilon}K_*^h(V'R) \cong {}_{-\varepsilon}K_{*+1}^h(U'R). \quad \square$$

Remark 2.3.5. Using Lemma 2.1.11, the Theorem 2.3.3 is equivalent to the statement that θ_0 induces an isomorphism

$$\theta_0 \star - : K_*^h(R) \cong {}_{-1}K_{*+2}^h((U')^2R).$$

Corollary 2.3.6. *Let $A \in \text{Alg}_{\ell}^*$ and assume that ℓ satisfies the λ -assumption 1.1.24. The element $\theta = c_2(\theta_0) \in {}_{-1}KH_2^h((U')^2\mathbb{Z})$ induces an isomorphism*

$$\theta \star - : {}_{\varepsilon}KH_*^h(A) \rightarrow {}_{-\varepsilon}KH_{*+2}^h((U')^2A)$$

Proof. Using that K_n^h satisfies excision for $n \leq 0$ and Theorem 2.3.3 we get that for any $A \in \text{Alg}_{\ell}^*$, θ_0 induces an isomorphism

$$\theta_0 : {}_{\varepsilon}K_*^h(A) \cong {}_{-\varepsilon}K_{*+2}^h((U')^2A). \quad (* \leq -2)$$

This then follows from Corollary 2.2.8 upon taking colimits. □

The 12-term exact sequence

Definition 2.3.7. Let R be a unital $*$ -ring. The involution of R induces an involution $g \rightarrow (g^*)^{-1}$ in $\mathrm{GL}_\infty(R)$ which in turn induces a natural action of $\mathbb{Z}/2$ in $K_*(R)$; for $x \in K_n(R)$ write \bar{x} for this action. Define

$$\begin{aligned} {}_\varepsilon W_n(R) &:= \mathrm{coker}(K_n(R) \xrightarrow{(\phi_R)^*} {}_\varepsilon K_n^h(R)), \\ {}_\varepsilon W'_n(R) &:= \mathrm{ker}({}_\varepsilon K_n^h(R) \xrightarrow{(\eta_R)^*} K_n(R)), \\ k_n(R) &:= \{x \in K_n(R) : \bar{x} = x\} / \{x = y + \bar{y} \text{ for some } y\}, \text{ and} \\ k'_n(R) &:= \{x \in K_n(R) : \bar{x} = -x\} / \{x = y - \bar{y} \text{ for some } y\}. \end{aligned}$$

The groups ${}_\varepsilon W_n(R)$ and ${}_\varepsilon W'_n(R)$ are called the Witt and coWitt groups of R . The groups $k_n(R)$ and $k'_n(R)$ are the corresponding $\mathbb{Z}/2$ -Tate cohomology groups of $K_n(R)$.

Theorem 2.3.8 (Suite exacte des douze, Karoubi [Kar80, Theoreme 4.3]). *Assume ℓ satisfies the λ -assumption 1.1.24 and let $R \in \mathrm{Alg}_\ell^*$ be a unital $*$ -algebra. There is an exact sequence*

$$\begin{array}{ccccccccccc} k_{n+1}(R) & \longrightarrow & -{}_\varepsilon W_{n+2}(R) & \longrightarrow & {}_\varepsilon W'_n(R) & \longrightarrow & k'_{n+1}(R) & \longrightarrow & -{}_\varepsilon W'_{n+1}(R) & \longrightarrow & -{}_\varepsilon W_n(R) \\ \uparrow & & & & & & & & & & \downarrow \\ W_{n+1}(R) & \longleftarrow & W'_{n+1}(R) & \longleftarrow & k'_{n+1}(R) & \longleftarrow & -{}_\varepsilon W'_n(R) & \longleftarrow & W_{n+2}(R) & \longleftarrow & k_{n+1}(R) \end{array}$$

Chapter 3

Bivariant Hermitian Algebraic K -theory

In this chapter we construct the bivariant hermitian algebraic K -theory category and develop some of its basic properties. This construction is based on the original bivariant algebraic K -theory $j : \text{Alg}_\ell \rightarrow kk$ made by Cortiñas and Thom in [CT07]. There are generalizations of kk to incorporate the action of groups and group graded algebras [Ell14] and also for algebras with actions of quantum groups [Ell18]. In Section 3.1 we develop the necessary results to construct kk^h as a category and the functor $j^h : \text{Alg}_\ell^* \rightarrow kk^h$. Then in Section 3.2 we show it is triangulated and prove how $j^h : \text{Alg}_\ell^* \rightarrow kk^h$ is the universal excisive homology theory (defined in such section) with matrix and hermitian stability and homotopy invariance.

From this chapter on we will assume that ℓ satisfies the λ -assumption 1.1.24 without further mention.

3.1 The kk^h category

Fix an infinite set X . A bijection $X \amalg X \cong X$ induces a $*$ -homomorphism $\mathcal{M}_X \oplus \mathcal{M}_X \rightarrow \mathcal{M}_X$; write \boxplus for its ind- $*$ -homotopy class. By Lemma 1.3.6, \boxplus is independent of the choice of bijection above.

Lemma 3.1.1 ([cf. CT07, Section 4.1]). *The map \boxplus together with the zero map, makes \mathcal{M}_X an abelian monoid object in $\text{ind} - [\text{Alg}_\ell^*]$*

Proof. Since any chosen bijection $X \amalg X \cong X$ also induces a bijection $X \amalg X \amalg X \cong X$ in any possible association and these choices induce the same class in $\text{ind} - [\text{Alg}_\ell^*]$, it is clear that \boxplus is associative. Similarly, the permutation of copies of X in $X \amalg X$ induce the same isomorphism as \boxplus and therefore it is commutative.

Let $X_0, X_1 \subseteq X$ be the corresponding subsets to $X \amalg \emptyset$ and $\emptyset \amalg X$ through the bijection $X \amalg X \cong X$. Write $f_0 : X_0 \rightarrow X$ and $f_1 : X_1 \rightarrow X$ the corresponding

bijections. Then, we have

$$\begin{aligned} [(f_0 \amalg \emptyset)^*](\text{id} \boxplus 0) &= \text{id}_{\mathcal{M}_X} \\ [(\emptyset \amalg f_1)^*](0 \boxplus \text{id}) &= \text{id}_{\mathcal{M}_X}. \end{aligned}$$

Therefore the zero map is a neutral element for \boxplus . \square

Similarly, any choice of bijection $X \times X \cong X$ gives rise to the same ind- $*$ -homotopy class of a $*$ -homomorphism $\mathcal{M}_X \otimes \mathcal{M}_X \rightarrow \mathcal{M}_X$; we write μ for this ind- $*$ -homotopy class.

Lemma 3.1.2. *The map μ is an associative and commutative product in $\text{ind} - [\text{Alg}_\ell^*]$ and the inclusion $i : \ell \rightarrow \mathcal{M}_X$ is an identity map for μ . Furthermore μ distributes over \boxplus and therefore $(\mathcal{M}_X, \boxplus, \mu, 0, [i])$ is a semi-ring object in $\text{Ind} - [\text{Alg}_\ell^*]$.*

Proof. Associativity and commutativity are proven in the same way as in the previous lemma and it is clear that i is an identity for μ . Finally, the fact that μ distributes over \boxplus can be derived from the fact that there is a natural bijection

$$X \times (X \amalg X) \cong (X \times X) \amalg (X \times X)$$

and Lemma 1.3.6. \square

Let $A, B \in \text{ind} - \text{Alg}_\ell^*$. Put

$$\{A, B\} := [A, \mathcal{M}_X B]; \quad (3.1.3)$$

the monoid operation \boxplus on \mathcal{M}_X induces one on $\{A, B\}$.

Lemma 3.1.4. *The product μ induces a bilinear, associative composition law:*

$$\begin{aligned} \star : \{B, C\} \times \{A, B\} &\rightarrow \{A, C\} \\ ([f], [g]) &\mapsto [\mu \otimes \text{id}_C] \circ [(\text{id}_{\mathcal{M}_X} \otimes f)] \circ [g]. \end{aligned}$$

Proof. Since changing the representative of the class $[f]$ does not change the class of $[\text{id}_{\mathcal{M}_X} \otimes f]$, it is clear that \star is well defined. The fact that \star is bilinear follows from the fact that μ distributes over \boxplus . Finally associativity follows from observing that for any map $h : C \rightarrow \mathcal{M}_X D$, the diagram

$$\begin{array}{ccccc} \mathcal{M}_X \mathcal{M}_X C & \xrightarrow{\text{id}_{\mathcal{M}_X} \otimes \text{id}_{\mathcal{M}_X} \otimes h} & \mathcal{M}_X \mathcal{M}_X \mathcal{M}_X D & \xrightarrow{\text{id}_{\mathcal{M}_X} \otimes \mu \otimes \text{id}_D} & \mathcal{M}_X \mathcal{M}_X D \\ \mu \otimes \text{id}_D \downarrow & & & & \downarrow \mu \otimes \text{id}_D \\ \mathcal{M}_X C & \xrightarrow{\text{id}_{\mathcal{M}_X} \otimes h} & \mathcal{M}_X \mathcal{M}_X D & \xrightarrow{\mu \otimes \text{id}_D} & \mathcal{M}_X D \end{array}$$

commutes due to the associativity of μ . \square

Definition 3.1.5. Let $\{\text{ind} - \text{Alg}_\ell^*\}_X$ be the category with the same objects as $\text{ind} - \text{Alg}_\ell^*$, where morphisms sets are given by (3.1.3) and which is enriched over the category of abelian monoids. Lemma 3.1.2 also shows that for $A \in \text{ind} - \text{Alg}_\ell^*$ the inclusion $i : A \rightarrow \mathcal{M}_X A$ is the identity. Write $\{\text{Alg}_\ell^*\}_X$ for full subcategory of $\{\text{ind} - \text{Alg}_\ell^*\}_X$ where the objects are in Alg_ℓ^* instead of $\text{ind} - \text{Alg}_\ell^*$.

Remark 3.1.6. Let $A, B \in \text{Alg}_\ell^*$. The algebra B^Δ has natural binary operation called concatenation $\bullet : B^\Delta \times B^\Delta \rightarrow B^\Delta$; this induces a binary operation in $[A, B^{\mathbb{S}^1}]$: for maps $f, g : A \rightarrow B^{\mathbb{S}^1}$ we write $f \bullet g$ for their concatenation. The zero map is a neutral element for this operation and the reversing map

$$\begin{aligned} \ell[t] &\rightarrow \ell[t] \\ t &\mapsto 1 - t \end{aligned} \tag{3.1.7}$$

induces a $*$ -morphism $a : B^{\mathbb{S}^1} \rightarrow B^{\mathbb{S}^1}$ such that $[f \bullet af] = [0]$. Concatenation and \boxplus distribute over each other in $\{A, B^{\mathbb{S}^1}\}$ [see CT07, Section 3.3].

Lemma 3.1.8. *Let $A, B \in \text{Alg}_\ell^*$. For $n \geq 1$, the concatenation and \boxplus operations coincide in $\{A, B^{\mathbb{S}^n}\}$ and it is an abelian group with such operation.*

Proof. As said in Remark 3.1.6, \bullet and \boxplus distribute over each other, due to the Eckmann-Hilton argument, both operations coincide. Since concatenation has an inverse as discussed in the same remark, the abelian monoid $\{A, B^{\mathbb{S}^n}\}$ is a group. \square

There is a canonical functor $[\text{Alg}_\ell^*] \rightarrow \{\text{Alg}_\ell^*\}$, which is the identity on objects and sends the class of a map f to that of if .

Lemma 3.1.9. *The composite functor $\text{can} : \text{Alg}_\ell^* \rightarrow [\text{Alg}_\ell^*] \rightarrow \{\text{Alg}_\ell^*\}$ is homotopy invariant, M_X -stable and i_+ -stable. Moreover any functor $H : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ which is homotopy invariant M_X -stable and i_+ -stable, factors uniquely through can .*

Proof. Since can factors through $[\text{Alg}_\ell^*]$, it is homotopy invariant by definition. Moreover for any functor H as in the statement, H factors through $\text{Alg}_\ell^* \rightarrow [\text{Alg}_\ell^*]$.

To see M_X stability, for any $x \in X$, the inclusions $i_{x,A} : A \rightarrow M_X A$ maps to $i_x, A : A \rightarrow M_X A$ in $\{\text{Alg}_\ell^*\}$. The identity map $M_X A \rightarrow M_X A$ induces a map $M_X A \rightarrow A$ in $\{\text{Alg}_\ell^*\}$ using the isomorphism $M_X M_X \cong M_X$. It is immediate that these maps are inverses to each other. can induces the identity in $\{\text{Alg}_\ell^*\}$ so can is M_X -stable. Similarly, since the ind-system \mathcal{M}_X is built with repeated composition of i_+ , using Lemma 1.2.5 we get that it is i_+ stable.

Finally, for a functor H as in the statement of the lemma, as said before H factors through $[H] : [\text{Alg}_\ell^*] \rightarrow \mathfrak{C}$. Since H is M_X -stable and i_+ -stable, for any $B \in \text{Alg}_\ell^*$, the map $[H](i_B : B \rightarrow \mathcal{M}_X B)$ is an isomorphism in \mathfrak{C} , so we can define

$$\{H\}([f : A \rightarrow \mathcal{M}_X B]) = [H](i_B)^{-1} \circ [H]([f]).$$

It is easy to see that $\{H\}$ defines a functor $\{H\} : \{\text{Alg}_\ell^*\} \rightarrow \mathfrak{C}$ that factors H through can . \square

Lemma 3.1.10. *The canonical functor $\text{can} : \text{Alg}_\ell^* \rightarrow \{\text{Alg}_\ell^*\}$ is hermitian stable.*

Proof. Since ℓ satisfies the λ -assumption 1.1.24, the proof follows from Lemma 3.1.9 and Corollary 1.6.16. \square

Lemma 3.1.11. *Let R be a unital $*$ -algebra, $A \trianglelefteq R$ a $*$ -ideal and $\lambda_1, \lambda_2 \in R$ be central elements satisfying the requirements of the element λ in the λ -assumption 1.1.24. Let*

$$p_i = p_{\lambda_i} = \begin{pmatrix} \lambda_i^* & 1 \\ \lambda_i \lambda_i^* & \lambda_i \end{pmatrix}$$

and let $\iota_i : A \rightarrow {}_1M_2A$, $\iota_i(a) = p_i a$. Then $\text{can}(\iota_1) = \text{can}(\iota_2)$ is an isomorphism in $\{\text{Alg}_\ell^*\}$.

Proof. Let $u_i = u_{\lambda_i}$ be as in (1.1.27) of Remark 1.1.26. Under the isomorphism ${}_1M_2 \cong M_\pm$, ι_i corresponds to ι_+ . Thus $\text{can}(\iota_i)$ is an isomorphism. Moreover, since $u = u_2 u_1^{-1} \in {}_1M_2R$ is unitary, $\text{can}(\text{ad}(u)) = \text{id}_{{}_1M_2A}$ by Lemma 1.6.8, we get

$$\text{can}(\iota_2) = \text{can}(\text{ad}(u_2 u_1^{-1})) \text{can}(\iota_1) = \text{can}(\iota_1). \quad \square$$

Lemma 3.1.12. *The functor $J : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell^*$ passes down to a functor $J : \{\text{Alg}_\ell^*\} \rightarrow \{\text{Alg}_\ell^*\}$.*

Proof. For a map $[f] \in [A, B]$, it is easy to check using the universal extension that the class $[J(f)] \in [J(A), J(B)]$ does not depend on the representative of the class f .

Recall the map $\phi_{M_X, B} : J(M_X B) \rightarrow M_X J(B)$ from Example 1.4.12. This induces a map $[\phi] \in \{J(\mathcal{M}_X B), B\}$. For a map $\xi = [f] \in \{A, B\}$, define $J(\xi) \in \{A, B\}$ as the class of the composition

$$J(A) \xrightarrow{J(f)} J(\mathcal{M}_X B) \xrightarrow{\phi} M_X J(B).$$

Using Remark 1.4.11, it is clear that this defines a functor. \square

From here on, we shall abuse notation and use the same letter for the homotopy class of a map $f : A \rightarrow B \in \text{ind} - \text{Alg}_\ell^*$ and for its image in $\{A, B\}$, and in case the latter is an abelian group (e.g. if $B = C^{\mathbb{S}^n}$) we put $-f$ for the inverse of $\text{can}(f)$ in that group.

Lemma 3.1.13. *Let $A, B \in \text{ind} - \text{Alg}_\ell^*$ and $f \in [A, B]$. The square*

$$\begin{array}{ccc} J(A) & \xrightarrow{J(f)} & J(B) \\ \downarrow \rho_A & & \downarrow \rho_B \\ A^{\mathbb{S}^1} & \xrightarrow{\text{id}_{\mathbb{S}^1} \otimes f} & B^{\mathbb{S}^1} \end{array}$$

is homotopy commutative.

Proof. This is direct consequence of Remark 1.4.11. \square

Lemma 3.1.14 ([cf. CMR07, Lemma 6.30]). *Let $A \in \text{Alg}_\ell^*$. Recall the maps $\rho_A : J(A) \rightarrow \Omega A$ and $\gamma_A : J(A^{\mathbb{S}^1}) \rightarrow J(A)^{\mathbb{S}^1}$ from (1.4.13) and (1.4.15) respectively. Then the following diagram commutes in $\{\text{ind} - \text{Alg}_\ell^*\}$.*

$$\begin{array}{ccc} J^2(A) & \xrightarrow{-\rho_{JA}} & J(A)^{\mathbb{S}^1} \\ J(\rho_A) \downarrow & \nearrow \gamma_A & \\ J(A^{\mathbb{S}^1}) & & \end{array}$$

Proof. Recall the reversing map $a : \ell[t] \rightarrow \ell[t]$ from (3.1.7). For an element $p \in \ell[t]$, observe that $\text{ev}_0(p) = \text{ev}_1(a(p))$ and $\text{ev}_1(p) = \text{ev}_0(a(p))$. Writing $P'\ell = \ker \text{ev}_1$, we get that $a(P\ell) = P'\ell$ and $a(P'\ell) = P\ell$. Passing to the subdivision versions, for any $A \in \text{Alg}_\ell^*$, a induces an isomorphism $\mathcal{P}A \cong \mathcal{P}'A$. Observe as well that since $P\ell \cap P'\ell = \Omega$ then $\ker(\text{ev}_0 : \mathcal{P}'A \rightarrow A) \cong A^{\mathbb{S}^1}$.

Define

$$I = \ker(\mathcal{P}T(A) \xrightarrow{\text{ev}_1} T(A) \xrightarrow{\eta_A} A) = \{p \in \mathcal{P}T(A) : \text{ev}_1(p) \in J(A)\}$$

and E as the pullback of the diagram

$$\begin{array}{ccc} E & \xrightarrow{pr_2} & I \\ \downarrow pr_1 & \lrcorner & \downarrow \text{ev}_1 \\ \mathcal{P}'J(A) & \xrightarrow{\text{ev}_0} & J(A). \end{array}$$

The surjection $pr_2 : E \rightarrow I$ is semi-split by $p \mapsto (p, t \text{ev}_0(p))$ and its kernel is

$$\{(q, 0) \in E : \text{ev}_0(q) = 0\} \cong \ker \text{ev}_0(\mathcal{P}'J(A) \rightarrow J(A)) \cong J(A)^{\mathbb{S}^1}.$$

Therefore, there is a semi-split extension

$$0 \rightarrow J(A)^{\mathbb{S}^1} \xrightarrow{i_1} E \xrightarrow{pr_2} I \rightarrow 0. \quad (3.1.15)$$

Also, by definition of I , there is a semi-split extension

$$0 \rightarrow I \rightarrow \mathcal{P}T(A) \xrightarrow{\eta_A \text{ev}_1} A \rightarrow 0. \quad (3.1.16)$$

Therefore there are maps of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \xrightarrow{i_1} & E & \xrightarrow{pr_2} & I \longrightarrow 0 \\ & & \parallel & & \downarrow pr_1 & & \downarrow \text{ev}_1 \\ 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \longrightarrow & \mathcal{P}'J(A) & \xrightarrow{\text{ev}_0} & J(A) \longrightarrow 0. \end{array} \quad (3.1.17)$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{P}T(A) & \xrightarrow{\eta_A \text{ev}_1} & A \longrightarrow 0 \\ & & \downarrow \text{ev}_1 & & \downarrow \text{ev}_1 & & \parallel \\ 0 & \longrightarrow & J(A) & \longrightarrow & T(A) & \xrightarrow{\eta_A} & A \longrightarrow 0. \end{array} \quad (3.1.18)$$

Let $\xi : J(I) \rightarrow J(A)^{\mathbb{S}^1}$ be the classifying map of the extension (3.1.15). Using Remark 1.4.11 and Remark 3.1.6, it follows that the classifying map of the bottom row of (3.1.17) is $-\rho_{J(A)} : J^2(A) \rightarrow J(A)^{\mathbb{S}^1}$. So from Remark 1.4.11 it follows that the map of extensions (3.1.17) gives the equality

$$\xi = \text{id}_{J(A)^{\mathbb{S}^1}} \circ \xi = -\rho_{J(A)} \circ J(\text{ev}_1). \quad (3.1.19)$$

Similarly let $\zeta : J(A) \rightarrow I$ the classifying map of extension (3.1.16); the map of extensions (3.1.18) gives

$$\text{ev}_1 \circ \zeta = \text{id}_{J(A)} \quad (3.1.20)$$

On the other hand, write $\bar{\eta} : I \rightarrow A^{\mathbb{S}^1}$ for the restriction of the map $(\text{id}_{\mathcal{P}} \otimes \eta_A) \bullet 0 : \mathcal{P}T(A) \rightarrow \mathcal{P}A$ (where \bullet is concatenation). The map $\bar{\eta}$ lifts to a map $q : E \rightarrow T(A)^{\mathbb{S}^1}$ by concatenation of paths in I and paths in $\mathcal{P}'J(A)$ which by definition of E they coincide in the endpoints. This gives maps of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \xrightarrow{i_1} & E & \xrightarrow{pr_2} & I \longrightarrow 0 \\ & & \parallel & & \downarrow q & & \downarrow \bar{\eta} \\ 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \longrightarrow & T(A)^{\mathbb{S}^1} & \xrightarrow{\text{id}_{\mathbb{S}^1} \otimes \eta_A} & A^{\mathbb{S}^1} \longrightarrow 0. \end{array} \quad (3.1.21)$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{P}T(A) & \xrightarrow{\eta_A \text{ ev}_1} & A \longrightarrow 0 \\ & & \downarrow \bar{\eta} & & \downarrow \text{id}_{\mathcal{P}} \otimes \eta & & \parallel \\ 0 & \longrightarrow & A^{\mathbb{S}^1} & \longrightarrow & \mathcal{P}A & \xrightarrow{\text{ev}_1} & A \longrightarrow 0. \end{array} \quad (3.1.22)$$

Since the classifying map of the bottom row of (3.1.21) is $\gamma_A : J(A^{\mathbb{S}^1}) \rightarrow J(A)^{\mathbb{S}^1}$, we get

$$\xi = \text{id}_{J(A)^{\mathbb{S}^1}} \circ \xi = \gamma_A \circ J(\bar{\eta}). \quad (3.1.23)$$

Also, since the classifying map of the bottom row of (3.1.22) is ρ_A , we have

$$\rho_A = \rho_A \circ \text{id}_A = \bar{\eta} \circ \zeta \quad (3.1.24)$$

Using (3.1.19),(3.1.20),(3.1.23) and (3.1.24):

$$\begin{aligned} \gamma_A \circ J(\rho_A) &= \gamma_A \circ J(\bar{\eta}) \circ J(\zeta) \\ &= \xi \circ J(\zeta) \\ &= -\rho_{J(A)} \circ J(\text{ev}_1) \circ J(\zeta) \\ &= -\rho_{J(A)} \end{aligned} \quad \square$$

Remark 3.1.25. The analogue of Lemma 3.1.14 for algebras without involution also holds as stated (this will be later deduced from the fact that kk is equivalent kk^h for a particular choice of ℓ). This corrects a mistake in [CT07, Lemma 6.2.2], where the sign is missing. A sign is also missing in the definition of composition in the category kk [CT07, Theorem 6.2.3], which is fixed below.

Let $A, B \in \text{Alg}_\ell^*$. As in [CT07, Section 6.1], using the functor $J : \{\text{Alg}_\ell^*\} \rightarrow \{\text{Alg}_\ell^*\}$ of Lemma 3.1.12, there is a map

$$\begin{aligned} \{A, B\} &\rightarrow \{JA, B^{\mathbb{S}^1}\} \\ \xi &\mapsto \rho_B \star J(\xi). \end{aligned}$$

Thus one can form the colimit

$$kk^h(A, B) = kk^h(A, B) = \underset{n}{\text{colim}} \{J^n A, B^{\mathbb{S}^n}\}$$

Lemma 3.1.26. *Let $\xi = [f] \in \{J^m B, C^{\mathbb{S}^m}\}$ and $\eta = [g] \in \{J^n A, B^{\mathbb{S}^n}\}$; put*

$$\xi \circ \eta = [(\text{id}_{\mathbb{S}^n} \otimes f)] \star (-1)^{mn} [\gamma_B^{m,n}] \star [J^m(g)] \in \{J^{m+n}(A), C^{\mathbb{S}^{m+n}}\}.$$

This defines a bilinear composition law

$$\begin{aligned} kk^h(B, C) \otimes_{\mathbb{Z}} kk^h(A, B) &\rightarrow kk^h(A, C) \\ \xi \otimes \eta &\mapsto \xi \circ \eta \end{aligned}$$

Proof. This follows from Lemma 3.1.13 and Lemma 3.1.14. \square

Therefore, the sets $kk^h(-, -)$ are the morphism sets of a category kk^h with the same objects as Alg_{ℓ}^* , where the identity map of $A \in Alg_{\ell}^*$ is represented by the class of $i : A \rightarrow \mathcal{M}_X A$. Define a functor $\{Alg_{\ell}^*\} \rightarrow kk^h$ as the identity on objects and as the canonical map to the colimit $\{A, B\} \rightarrow kk^h(A, B)$ on arrows. Composing the latter with the functor $Alg_{\ell}^* \rightarrow \{Alg_{\ell}^*\}$ we obtain a functor

$$j^h : Alg_{\ell}^* \rightarrow kk^h. \quad (3.1.27)$$

The category kk^h together with the functor j^h is called *bivariant algebraic hermitian K-theory*. We will often use the term *kk^h -equivalence* between two $*$ -algebras to mean that their corresponding images in kk^h are isomorphic.

3.2 j^h as an excisive homology theory

A triangulated category is a triple $(\mathfrak{T}, \Omega_{\mathfrak{T}}, \mathcal{T})$ where \mathfrak{T} is an additive category, $\Omega_{\mathfrak{T}} : \mathfrak{T} \rightarrow \mathfrak{T}$ is a self-equivalence functor called the loop functor and \mathcal{T} is a class of sequences of morphisms in \mathfrak{T}

$$\Omega_{\mathfrak{T}} C \rightarrow A \rightarrow B \rightarrow C$$

called (*distinguished*) *triangles* such that they satisfy the following axioms:

TR0 The class \mathcal{T} is closed under isomorphisms and the sequence

$$\Omega_{\mathfrak{T}} A \rightarrow 0 \rightarrow A \xrightarrow{\text{id}_A} A$$

is a distinguished triangle.

TR1 For any map $\alpha : A \rightarrow B$ in \mathfrak{T} , there is a distinguished triangle

$$\Omega_{\mathfrak{T}} B \rightarrow C \rightarrow A \xrightarrow{\alpha} B.$$

TR2 For the sequences

$$\Omega_{\mathfrak{T}} C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C, \quad (3.2.1)$$

$$\Omega_{\mathfrak{T}} B \xrightarrow{-\Omega_{\mathfrak{T}} h} \Omega_{\mathfrak{T}} C \xrightarrow{f} A \xrightarrow{g} B \quad (3.2.2)$$

one is a distinguished triangle if and only if the other is. In this case we say that (3.2.2) is a rotation of (3.2.1).

TR3 For any commutative diagram between distinguished triangles

$$\begin{array}{ccccccc} \Omega_{\mathfrak{T}}C & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ \downarrow \Omega_{\mathfrak{T}}\gamma & & & & \downarrow \beta & & \downarrow \gamma \\ \Omega_{\mathfrak{T}}C' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

there exists a map $\alpha : A \rightarrow A'$ which makes the whole diagram commute.

TR4 Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be maps in \mathfrak{T} . There is a commutative diagram

$$\begin{array}{ccccccc} \Omega_{\mathfrak{T}}^2C & \longrightarrow & \Omega_{\mathfrak{T}}D & \longrightarrow & \Omega_{\mathfrak{T}}B & \xrightarrow{\Omega_{\mathfrak{T}}\beta} & \Omega_{\mathfrak{T}}C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D''' & \xlongequal{\quad} & D''' & \longrightarrow & 0 \\ \downarrow & & \downarrow h & & \downarrow & & \downarrow \\ \Omega_{\mathfrak{T}}C & \xrightarrow{j} & D'' & \longrightarrow & A & \xrightarrow{\beta\alpha} & C \\ \parallel & & \downarrow & & \downarrow \alpha & & \parallel \\ \Omega_{\mathfrak{T}}C & \longrightarrow & D' & \longrightarrow & B & \xrightarrow{\beta} & C \end{array}$$

in which each row and column is a distinguished triangle. Furthermore, the square

$$\begin{array}{ccc} \Omega_{\mathfrak{T}}B & \xrightarrow{\Omega_{\mathfrak{T}}\beta} & \Omega_{\mathfrak{T}}C \\ \downarrow & & \downarrow j \\ D''' & \xrightarrow{h} & D'' \end{array}$$

commutes.

Remark 3.2.3. Usually the axioms for triangulated categories are defined using the inverse to the loop functor, called the *suspension* functor. In this thesis we present the axiom in this way since it will be more natural to work with the loop functor.

Let \mathfrak{T} be a triangulated category; write $[n]$ for the n -fold loop functor in \mathfrak{T} . Let \mathcal{E} be the class of all semi-split extensions

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0. \quad (E)$$

An *excisive homology theory* on Alg_{ℓ}^* (with coefficients in \mathfrak{T}) is a functor $H : \text{Alg}_{\ell}^* \rightarrow \mathfrak{T}$ together with a family of maps

$$\{\partial_E : H(C)[1] \rightarrow H(A) : E \in \mathcal{E}\}$$

such that for every $E \in \mathcal{E}$, the sequence

$$H(C)[1] \xrightarrow{\partial_E} H(A) \rightarrow H(B) \rightarrow H(C)$$

is a triangle in \mathfrak{T} and the maps $\{\partial_E\}$ are compatible with maps of extensions in the sense that for a commutative diagram between semi-split extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 & (E) \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 & (E') \end{array}$$

the following diagram

$$\begin{array}{ccc} H(C)[1] & \xrightarrow{\partial_E} & H(A) \\ \downarrow H(f_3) & & \downarrow H(f_1) \\ H(C')[1] & \xrightarrow{\partial_{E'}} & H(A') \end{array}$$

commutes.

Remark 3.2.4. In a triangulated category, a sequence

$$\Omega C \rightarrow A \rightarrow B \xrightarrow{f} C$$

with a splitting $g : C \rightarrow B$ (i.e. $\text{id}_C = fg$), is always isomorphic to the split distinguished triangle

$$\Omega C \xrightarrow{0} A \xrightarrow{i_1} A \oplus C \xrightarrow{pr_2} C.$$

In particular, the first sequence is a distinguished triangle [Nee01, Remark 1.2.7].

In what follows we will see that there is a natural triangulation of kk^h which makes the functor j^h a homology theory.

Lemma 3.2.5. *Let $L \in \text{Alg}_\ell^*$ be flat as an ℓ -module. The functor $L = L \otimes - : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell^*$ induces a functor $L : kk^h \rightarrow kk^h$.*

Proof. Using the universal property described in Lemma 3.1.9, the functor descends to $\{L\} : \{\text{Alg}_\ell^*\} \rightarrow \{\text{Alg}_\ell^*\}$.

Next, recall the map

$$\phi_{A,L} : J(L \otimes A) \rightarrow L \otimes J(A)$$

from Example 1.4.12. Write ϕ_L^n for the composition

$$J^n(L \otimes A) \xrightarrow{J^{n-1}(\phi_{A,L})} J^{n-1}(L \otimes J(A)) \rightarrow \dots \xrightarrow{\phi_{J^{n-1}(A),L}} L \otimes J^n(A).$$

For a map $\alpha \in kk^h(A, B)$ represented by $[f : J^n(A) \rightarrow \mathcal{M}_X B^{\mathbb{S}^n}]$ define $L \otimes \alpha \in kk^h(L \otimes A, L \otimes B)$ as the class of the composition

$$J^n(L \otimes A) \xrightarrow{\phi_L^n} L \otimes J^n(A) \xrightarrow{L \otimes f} L \otimes \mathcal{M}_X B^{\mathbb{S}^n} \cong \mathcal{M}_X(L \otimes B)^{\mathbb{S}^n}.$$

Using Remark 1.4.11, it is clear that this definition gives a functor $L : kk^h \rightarrow kk^h$. \square

Corollary 3.2.6 ([cf. CT07, Section 6.6]). *The functors $\Omega, \Sigma_X : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell^*$ induce functors $\Omega, \Sigma_X : kk^h \rightarrow kk^h$*

Proof. This follows from the previous lemma since Ω and Σ_X are flat. \square

Lemma 3.2.7. *Let $f : A \rightarrow B$ be semi-split $*$ -morphism. Then, for any subdivision $\mathcal{P}_{n,f} = B^{\text{sd}^n \Delta^1} \times_B A$ of the path algebra, the inclusion $i_f : \ker(f) \rightarrow \mathcal{P}_{n,f}$ induced by the inclusion $i_f : \ker(f) \rightarrow P_f$ and the last vertex map is invertible in kk^h .*

Proof. The same proof as in [CT07, Lemma 6.3.2] in the non-hermitian case works verbatim. \square

Corollary 3.2.8. *The last vertex map $h : \Omega A \rightarrow A^{\text{sd}^n S^1}$ is invertible in kk^h ; it follows that in the ind-object A^{S^1} all the transition maps are kk^h -equivalences.*

Proof. This follows from Lemma 3.2.7 by considering the loop extension (1.4.7) and that if $\mathcal{P}^n A$ is the n -th subdivision of PA then the kernel of the induced map $\text{ev}_1^n : \mathcal{P}^n A \rightarrow A$ is isomorphic to $A^{\text{sd}^n S^1}$ and that $P_{\text{ev}_1^n}$ is $\mathcal{P}_{n,\text{ev}_1}$. \square

Definition 3.2.9. For a semi-split extension

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \quad (E)$$

the Lemma 3.2.7 gives an kk^h -equivalence $A \rightarrow P_g$. Define the *connecting map* as the following morphism in $kk^h(\Omega C, A)$:

$$\partial_E : \Omega C \rightarrow P_g \xleftarrow{\sim} A, \quad (3.2.10)$$

the composition of the natural map $\Omega C \rightarrow P_g$ of the mapping path extension of Example 1.4.5 and the inverse of $A \xrightarrow{\sim} P_g$.

Lemma 3.2.11. *For a semi-split extension (E), the sequences*

$$\begin{aligned} &kk^h(D, \Omega B) \xrightarrow{\Omega j^h(g)_*} kk^h(D, \Omega C) \xrightarrow{(\partial_E)_*} kk^h(D, A) \xrightarrow{j^h(f)_*} kk^h(D, B) \xrightarrow{j^h(g)_*} kk^h(D, C) \\ &kk^h(C, D) \xrightarrow{j^h(g)^*} kk^h(B, D) \xrightarrow{j^h(f)^*} kk^h(A, D) \xrightarrow{(\partial_E)^*} kk^h(\Omega C, D) \xrightarrow{\Omega j^h(g)^*} kk^h(\Omega B, D) \end{aligned}$$

are exact.

Proof. This is proved in [CT07, Theorem 6.3.6 and Theorem 6.3.7] in the non-hermitian case. The same proof works verbatim. \square

Corollary 3.2.12. *For any $D \in \text{Alg}_\ell^*$, the functors*

$$kk^h(D, -), kk^h(-, D) : \text{Alg}_\ell^* \rightarrow \mathfrak{Ab}$$

are split exact.

Proof. This is [CT07, Corollary 6.3.4] in the non-hermitian case; again, the same proof works. \square

For $R \in \text{Alg}_\ell^*$ unital, the $*$ -algebra $\Gamma_X R$ is what is known as a $*$ -infinite-sum algebra: define

$$\alpha = \sum_{n \in \mathbb{N}} e_{n,2n} \text{ and } \beta = \sum_{n \in \mathbb{N}} e_{n,2n+1};$$

these elements satisfy the identities

$$\begin{aligned} \alpha^* \alpha &= 1 = \beta^* \beta \\ \alpha \alpha^* + \beta \beta^* &= 1. \end{aligned}$$

For $a, b \in \Gamma_X R$, define

$$\begin{aligned} a \oplus b &= \alpha^* a \alpha + \beta^* b \beta \\ a^\infty &= \sum_{n \in \mathbb{N}} (\beta^*)^n \alpha^* a \alpha \beta^n, \end{aligned}$$

and for $f, g : B \rightarrow \Gamma_X R$, write $f \oplus g : B \rightarrow \Gamma_X R$ and $f^\infty : B \rightarrow \Gamma_X R$ for

$$\begin{aligned} f \oplus g(b) &= f(b) \oplus g(b) \\ f^\infty(b) &= f(b)^\infty. \end{aligned}$$

Then, it is straightforward to compute that

$$\text{id}_{\Gamma_X R} \oplus \text{id}_{\Gamma_X R}^\infty = \text{id}_{\Gamma_X R}.$$

Lemma 3.2.13. *There exists a unitary matrix $Q \in M_3 \Gamma_X R$ such that for any $a, b \in \Gamma_X A$*

$$Q^* \begin{pmatrix} a \oplus b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof. The matrix Q in [Wag72, p.355] can easily be seen to be unitary in our case. \square

Corollary 3.2.14. *For any $A \in \text{Alg}_\ell^*$, the $*$ -algebra $\Gamma_X A$ is isomorphic to 0 in kk^h .*

Proof. Assume A unital, the general case follows from split-exactness. From Lemma 3.2.13, Lemma 1.6.8 and Lemma 1.6.9 we get that

$$j^h(\text{id}_{\Gamma_X A} \oplus \text{id}_{\Gamma_X A}^\infty) = j^h(\text{id}_{\Gamma_X A}) + j^h(\text{id}_{\Gamma_X A}^\infty);$$

since $\text{id}_{\Gamma_X A} \oplus \text{id}_{\Gamma_X A}^\infty = \text{id}_{\Gamma_X A}$, it follows that $j^h(\text{id}_{\Gamma_X A}) = 0$ and therefore $\Gamma_X A$ is kk^h -equivalent to 0. \square

Corollary 3.2.15. *There is a natural kk^h -equivalence $\Omega \Sigma_X A \cong A$. Since the functors Σ_X and Ω commute, it follows that they are inverse equivalences on kk^h .*

Proof. Write $q : \Gamma_X A \rightarrow \Sigma_X A$ for the quotient map. Using Lemma 3.2.7, there is a natural kk^h -equivalence $M_X A \cong P_q$. On other hand, considering the mapping path extension of q , there is a natural map $\Omega \Sigma_X A \rightarrow P_q$. Since $\Gamma_X A$ is kk^h -equivalent to 0 for any A , it follows from Lemma 3.2.11, that for any $D \in \text{Alg}_\ell^*$, the inclusion $\Omega \Sigma_X A \rightarrow P_q$ induces isomorphisms

$$kk^h(\Omega \Sigma_X A, D) \cong kk^h(P_q, D).$$

Therefore there are kk^h -equivalences $\Omega \Sigma A \cong P_q \cong M_X A \cong A$. \square

Lemma 3.2.16. *The classifying map $\rho_A : J(A) \rightarrow \Omega A$ is an kk^h -equivalence.*

Proof. The algebras $T(A)$ and PA are *contractible*: there are $*$ -homotopies $H_0 : T(A) \rightarrow T(A)[s]$ and $H_1 : PA \rightarrow PA[s]$ such that

$$\begin{aligned} \text{ev}_1 H_0 &= \text{id}_{T(A)} & \text{ev}_0 H_0 &= 0 \\ \text{ev}_1 H_1 &= \text{id}_{PA} & \text{ev}_0 H_1 &= 0. \end{aligned}$$

These are defined as follows: H_0 is the adjoint to the ℓ -linear map

$$\begin{aligned} A &\rightarrow T(A)[s] \\ a &\mapsto sa; \end{aligned}$$

similarly, H_1 is defined by

$$\begin{aligned} PA &\rightarrow PA[s] \\ p(t) &\mapsto p(st). \end{aligned}$$

Therefore, using the loop (1.4.7) and the universal extensions in Lemma 3.2.11, there are natural equivalences $p_{\text{loop}} : \Omega A \rightarrow \Omega A$ and $p_{\text{univ}} : \Omega A \rightarrow J(A)$. Using naturality of these maps and the map of extensions from the universal to the loop extension that defines ρ_A , the statement of the theorem follows. \square

Let \mathcal{T} be the class of sequences in kk^h

$$\Omega C \rightarrow A \rightarrow B \rightarrow C$$

which are isomorphic (as sequences) to the image of some mapping path extension

$$\Omega B' \rightarrow P_f \rightarrow A' \xrightarrow{f} B'.$$

Theorem 3.2.17. *The triple $(kk^h, \Omega, \mathcal{T})$ is a triangulated category.*

Proof. This is proved in [CT07, Theorem 6.5.2] for the non-hermitian case. The same proof works verbatim. \square

Theorem 3.2.18. *The functor $j^h : \text{Alg}_\ell^* \rightarrow kk^h$ together with the connecting maps $\{\partial_E\}$ form an excisive homology theory which is homotopy invariant and M_X and hermitian stable.*

Proof. The fact that j^h is homotopy invariant and M_X and hermitian stable follows from Lemma 3.1.9 and Lemma 3.2.11. By definition of the connecting map, for a semi-split extension (E) the sequence

$$\Omega C \xrightarrow{\partial_E} A \rightarrow B \rightarrow C$$

is isomorphic (as a sequence) to the mapping path triangle of the extension. Moreover, the maps ∂_E are clearly natural on the extension (E) . \square

Remark 3.2.19. Theorem 3.2.18 corrects an error in [CT07, Example 6.6.1] in which the connecting map is wrongly defined.

Theorem 3.2.20. *The functor $j^h : \text{Alg}_\ell^* \rightarrow kk^h$ is universal in the following sense: for any excisive homology theory $H : \text{Alg}_\ell^* \rightarrow \mathfrak{T}$ that is homotopy invariant, M_X and hermitian stable, there is a unique triangulated functor $\overline{H} : kk^h \rightarrow \mathfrak{T}$ such that following diagram commutes*

$$\begin{array}{ccc} \text{Alg}_\ell^* & \xrightarrow{H} & \mathfrak{T} \\ j^h \downarrow & \nearrow \overline{H} & \\ kk^h & & \end{array}$$

Proof. This is [CT07, Theorem 6.6.2] in the non-hermitian case. The same proof works. \square

Remark 3.2.21. As explained in Remark 1.4.4, the classes of extensions which are semi-split with respect to the underlying categories of sets and ℓ -modules agree with those semi-split with respect to sets with involution and ℓ -modules with involution. Hence by Theorem 3.2.20, the corresponding kk -theories are the same whether involutions are included in the underlying category or not.

Let \mathfrak{C} be an abelian category. A functor $H : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ is *half-exact* if for an extension in Alg_ℓ^*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

the sequence

$$H(A) \rightarrow H(B) \rightarrow H(C)$$

is exact.

Proposition 3.2.22. *Let \mathfrak{C} be an abelian category and $H : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ a functor. Assume that H is half-exact, homotopy invariant and M_X and hermitian stable. Then there is a unique homological functor $\overline{H} : kk^h \rightarrow \mathfrak{C}$ such that $\overline{H} \circ j^h = H$.*

Proof. Again, the proof is the same as in [CT07, Theorem 6.6.6]. \square

Remark 3.2.23. For a map $\alpha \in kk^h(A, B)$ we will show how to describe $\overline{H}(\alpha)$ for a functor H as in Proposition 3.2.22: first extend H to $\{H\}$ as in Lemma 3.1.9; next realize α as a class of a map $f : J^n(A) \rightarrow M_X M_\pm^{\otimes k} B^{\text{sd}^r S^n}$. Composing with the inverse of $J^n(A) \rightarrow \Omega^n A$ and using M_X -stability, hermitian stability and Corollary 3.2.8 we get a map $\overline{f} : \Omega^n A \rightarrow \Omega^n B$ in kk^h . It is immediate to see that \overline{f} induces the class of $\Omega^n(\alpha)$, and therefore $\overline{H}(\Omega^n(\alpha))$ is determined by $\{H\}(\overline{f})$ and in turn $\overline{H}(\alpha) = \{H\}(\Sigma_X^n \overline{f})$.

From here on, we will fix $X = \mathbb{N}$.

Remark 3.2.24. Let $f : A \rightarrow B$ be a semi-split $*$ -morphism. One can also fit f into other equivalent triangles instead of the one induced by P_f . For example, take the pullback of the natural map $T(B) \rightarrow B$ along f

$$\begin{array}{ccccc} J(B) & \longrightarrow & T(B) \times_B A & \longrightarrow & A \\ \downarrow & & \downarrow & \lrcorner & \downarrow f \\ J(B) & \longrightarrow & T(B) & \longrightarrow & B. \end{array}$$

Write $T_f := T(B) \times_B A$. Then, we have a commutative diagram

$$\begin{array}{ccccccc} J(B) & \longrightarrow & T_f & \longrightarrow & A & \xrightarrow{f} & B \\ \downarrow \rho_B & & \downarrow & & \parallel & & \parallel \\ \Omega B & \longrightarrow & P_f & \longrightarrow & A & \xrightarrow{f} & B \end{array} \quad (3.2.25)$$

By Lemma 3.2.16, the vertical map $JB \rightarrow \Omega B$ is a kk^h -equivalence. Since the first three terms of the top row in (3.2.25) form an extension, using the five lemma it follows that the vertical map $T_f \rightarrow P_f$ is a kk^h -equivalence. Thus, the top row is kk^h -isomorphic to the bottom row, and is thus a triangle in kk^h .

In a similar case, let Γ_f be as in Example 1.4.9. By Corollary 3.2.14, the classifying map $J\Sigma B \rightarrow M_\infty B$ of the cone extension is a kk^h -equivalence, and therefore $T_{\Sigma f} \rightarrow \Gamma_f$ is a kk^h -equivalence by the same reasoning as before. Thus the vertical maps in the commutative diagram below form an isomorphism of triangles in kk^h :

$$\begin{array}{ccccccc} J\Sigma B & \longrightarrow & T_{\Sigma f} & \longrightarrow & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \\ \downarrow & & \downarrow & & \parallel & & \parallel \\ M_\infty B & \longrightarrow & \Gamma_f & \longrightarrow & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \end{array} \quad (3.2.26)$$

Therefore, the bottom row of (3.2.26) is a distinguished triangle in kk^h . The map (3.2.26) together with that of (3.2.25) with $\Sigma(f)$ substituted for f is a zig-zag of kk^h -equivalences. In particular Γ_f is kk^h -equivalent to $P_{\Sigma f}$. Since ΣP_f is isomorphic to $P_{\Sigma f}$, the bottom row of (3.2.26) is isomorphic in kk^h to the suspension of the mapping path extension 1.4.9 associated to Σf . Thus, we have an isomorphism of triangles:

$$\begin{array}{ccccccc} M_\infty B & \longrightarrow & \Gamma_f & \longrightarrow & \Sigma A & \xrightarrow{j^h(\Sigma f)} & \Sigma B \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ B & \longrightarrow & \Sigma P_f & \longrightarrow & \Sigma A & \xrightarrow{\Sigma j^h(f)} & \Sigma B \end{array}$$

Remark 3.2.27. Let $(Alg_\ell^*)_f \subset Alg_\ell^*$ and $kk_f^h \subset kk^h$ be the full subcategories whose objects are the $*$ -algebras that are flat as ℓ -modules and let $j_f^h : (Alg_\ell^*)_f \rightarrow kk_f^h$ be

the restriction of j^h . Observe that $(\text{Alg}_\ell^*)_f$ is closed under J and under mapping path extensions; hence kk_f^h is triangulated and j_f^h is excisive, homotopy invariant, ι_+ -stable and M_X -stable. Moreover, in the same way as in Theorem 3.2.20, the functor j_f^h is universal among such functors.

Example 3.2.28. Let ℓ_0 be any commutative ring and let $\ell = \text{inv}(\ell_0)$ and $\text{inv} : \text{Alg}_{\ell_0} \rightarrow \text{Alg}_\ell^*$ be as in Example 1.1.6. Recall the universal excisive matrix stable and homotopy invariant homology theory $j : \text{Alg}_{\ell_0} \rightarrow kk$. Then, the composition $j^h \circ \text{inv} : \text{Alg}_\ell^* \rightarrow kk^h$ is excisive, homotopy invariant and M_X -stable; by universality of j it induces a triangulated functor $\text{inv} : kk_{\ell_0} \rightarrow kk_\ell^h$. Similarly, for the inverse functor to inv ,

$$\begin{aligned} \text{res} : \text{Alg}_\ell^* &\rightarrow \text{Alg}_{\ell_0} \\ B &\mapsto (1, 0)B \end{aligned}$$

the composition $j \circ \text{res}$ is excisive, homotopy invariant, M_X -stable and by Example 1.1.16 it is also hermitian stable. Hence it induces a functor $\text{res} : kk_\ell^h \rightarrow kk_{\ell_0}$ which is inverse to inv . This shows that kk is a particular case of kk^h .

Similarly, for an arbitrary ℓ , recall the adjunctions from Example 1.1.6:

$$\begin{aligned} \text{res} : \text{Alg}_\ell^* &\leftrightarrow \text{Alg}_\ell : \text{ind}, \\ \text{ind}' : \text{Alg}_\ell &\leftrightarrow \text{Alg}_\ell^* : \text{res}. \end{aligned}$$

The same reasoning as before gives adjunctions

$$\begin{aligned} \text{res} : kk^h &\leftrightarrow kk : \text{ind}, \\ \text{ind}' : kk &\leftrightarrow kk^h : \text{res}. \end{aligned}$$

Example 3.2.29. Let $L \in \text{Alg}_\ell^*$; then $L \otimes -$ preserves semi-split extensions with linear splittings if either L is flat as ℓ -module or every semi-split extension is ℓ -linearly split. In either case, $j^h(L \otimes -) : \text{Alg}_\ell^* \rightarrow kk^h$ is homotopy invariant, matricially stable, hermitian stable and excisive, and therefore induces a triangulated functor $L \otimes - : kk^h \rightarrow kk^h$. By a similar argument, for kk_f^h as in Remark 3.2.27, any $L \in \text{Alg}_\ell^*$ induces a triangulated functor $L \otimes - : kk_f^h \rightarrow kk_f^h$.

Proposition 3.2.30. *Let $A_1, A_2 \in \text{Alg}_\ell^*$ such that $A_i \otimes -$ ($i = 0, 1$) preserve linearly split extensions. Then we have a natural bilinear, associative product*

$$kk^h(A_1, A_2) \times kk^h(B_1, B_2) \rightarrow kk^h(A_1 \otimes B_1, A_2 \otimes B_2), \quad (\xi, \eta) \mapsto \xi \otimes \eta$$

that is compatible with composition in all variables.

Proof. Suppose first the case that A_1, A_2 are flat as ℓ -modules. By Example 3.2.29, $A_i \otimes -$ and $- \otimes B_i$ extend to functors $A_i \otimes - : kk^h \rightarrow kk^h$ and $- \otimes B_i : kk_f^h \rightarrow kk_f^h$. For $\xi \in kk^h(A_1, A_2)$ and $\eta \in kk^h(B_1, B_2)$, set

$$\xi \otimes \eta = (\xi \otimes \text{id}_{B_2}) \circ (\text{id}_{A_1} \otimes \eta).$$

It is straightforward to check that the product above has all the desired properties.

In the case semi-split extensions are always linearly split, then $- \otimes B_i$ extend to the functors $- \otimes B_i : kk^h \rightarrow kk^h$ and use the same definition as before. \square

Definition 3.2.31. Let $\varepsilon \in \ell$ be a unitary, $A, B \in \text{Alg}_\ell^*$ and $n \in \mathbb{Z}$. Put

$$kk_n^h(A, B) := \begin{cases} kk^h(A, \Sigma^n B) & \text{if } n \geq 0 \\ kk^h(A, \Omega^{-n} B) & \text{if } n < 0 \end{cases}$$

$${}_\varepsilon kk_n^h(A, B) := kk_n^h(A, {}_\varepsilon M_2 B)$$

Remark 3.2.32. Since Ω and Σ are inverse functors in kk^h there are natural isomorphisms

$$kk_n^h(A, B) \cong \begin{cases} kk^h(\Omega^n A, B) & \text{if } n \geq 0 \\ kk^h(\Sigma^{-n} A, B) & \text{if } n < 0 \end{cases}$$

Remark 3.2.33. Due to Remark 1.1.26, there is a $*$ -isomorphism ${}_1 M_2 \cong M_\pm$. It follows from this and from Theorem 3.2.20 that for all $A, B \in \text{Alg}_\ell^*$, $i_+ : \ell \rightarrow M_\pm$ induces a canonical isomorphism

$${}_1 kk_*^h(A, B) \cong kk_*^h(A, B).$$

Example 3.2.34. The functor $KH_0^h : \text{Alg}_\ell^* \rightarrow KH_0^h(\ell) - \text{Mod}$ satisfies the hypothesis of Proposition 3.2.22. Hence the functor \overline{KH}_0^h of the proposition induces a natural homomorphism

$$kk^h(A, B) \rightarrow \text{hom}_{KH_0^h(\ell)}(KH_0^h(A), KH_0^h(B))$$

Setting $A = \ell$ we obtain a natural map

$$kk^h(\ell, B) \rightarrow KH_0^h(B).$$

Proposition 3.2.35. *The product from Proposition 3.2.30 maps to the cup product from Lemma 2.2.9 under the map from Example 3.2.34. In other words, there is a commutative diagram*

$$\begin{array}{ccc} kk^h(\ell, A) \otimes_{\mathbb{Z}} kk^h(\ell, B) & \longrightarrow & KH_0^h(A) \otimes_{\mathbb{Z}} KH_0^h(B) \\ \downarrow \otimes & & \downarrow * \\ kk^h(\ell, A \otimes B) & \longrightarrow & KH_0^h(A \otimes B). \end{array}$$

Proof. Assume A, B unital and let $\alpha \in kk^h(\ell, A)$ and $\beta \in kk^h(\ell, B)$. Using Remark 3.2.23, the corresponding elements in $KH_0^h(A)$ and $KH_0^h(B)$ are determined by maps

$$\begin{aligned} \Omega^n(\alpha)_* &: KH_0^h(\Omega^n) \rightarrow KH_0^h(\Omega^n A), \\ \Omega^m(\beta)_* &: KH_0^h(\Omega^m) \rightarrow KH_0^h(\Omega^m B). \end{aligned}$$

and evaluation at $[1] \in KH_0^h(\ell)$. Since the product

$$kk^h(\ell, A) \otimes_{\mathbb{Z}} kk^h(\ell, B) \rightarrow kk^h(\ell, A \otimes B)$$

extends the tensor product of algebras and due to Remark 2.2.4 the cup product corresponds to the extension of scalars, it follows that

$$\Omega^{n+m}(\alpha \otimes \beta)_*[1] = \Omega^n(\alpha)_*[1] \star \Omega^m(\beta)_*[1].$$

From this, the statement follows in the unital case. The non-unital case follows from the unital one and excision. \square

Chapter 4

Computations and the comparison with KH^h

In this chapter we show some computations in kk^h as a matter of examples: in Section 4.1 we characterize the image of the coproducts, of the Toeplitz algebra and of the Cohn algebra of a graph and also give an algebraic analogue of the Pimsner-Voiculescu sequence. In Section 4.2 we show that the natural map $kk^h(\ell, A) \rightarrow KH_0^h(A)$ as described in Example 3.2.28 is an isomorphism.

4.1 Computations

Coproducts

Proposition 4.1.1. *Let $A, B \in \text{Alg}_\ell^*$. Then the natural map $A \amalg B \rightarrow A \oplus B$ is a kk^h -equivalence.*

Proof. Define $f : A \oplus B \rightarrow M_2(A \amalg B)$ as

$$f(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

We will show that

$$j^h(\text{id}_{M_2} \otimes \pi \circ f) = j^h(i_2 : A \oplus B \rightarrow M_2(A \oplus B)) \quad (4.1.2)$$

and that

$$j^h(f \circ \pi) = j^h(i_2 : \amalg B \rightarrow M_2(A \amalg B)); \quad (4.1.3)$$

it follows that $j^h(\pi)$ is left and right invertible and therefore an isomorphism in kk^h .

Identify $A \amalg B$ with its image through i_2 in $M_2(A \amalg B)$. Let $u(t) \in M_2(\widehat{A \amalg B}[t])$ defined by

$$u_t = \text{id}_{M_2 A} \amalg \begin{pmatrix} 1 - t^2 & t \\ t^3 - 2t & 1 - t^2 \end{pmatrix}.$$

It is easily shown that u_t is invertible, $u_0 = \text{id}$ and

$$\begin{aligned} u_1 i_2(a) u_1^* &= f \circ \pi(a) & (a \in A) \\ u_1 i_2(b) u_1^* &= f \circ \pi(b) & (b \in B); \end{aligned}$$

therefore, it follows from Lemma 1.2.3 that the equality in (4.1.3) stands. Similarly, using the matrix $\pi(u_t) \in M_2(\widetilde{A \oplus B})$ we can conclude the equality in (4.1.2). \square

Corollary 4.1.4. *The natural map $Q(A) \rightarrow A \oplus A$ is a kk^h -equivalence and it induces a kk^h -equivalence $\pi_0 : q(A) \rightarrow A$.*

Proof. This follows from Proposition 4.1.1 and the commutative diagram between split triangles in kk^h (which are distinguished by Remark 3.2.4):

$$\begin{array}{ccccccc} \Omega A & \xrightarrow{0} & q(A) & \longrightarrow & Q(A) & \longrightarrow & A \\ \parallel & & \downarrow \pi_0 & & \downarrow \wr & & \parallel \\ \Omega A & \xrightarrow{0} & A & \xrightarrow{j_1} & A \oplus A & \xrightarrow{pr_2} & A \end{array}$$

\square

The fundamental theorem

Recall from Example 1.1.21 the Laurent polynomial algebra $A[t, t^{-1}]$. Write

$$\sigma A = \ker(\text{ev}_1 : A[t, t^{-1}] \rightarrow A).$$

The *Toeplitz algebra* τ (over ℓ) is the $*$ -algebra generated by an element S such that $S^*S = 1$. We write τ_0 for the kernel of the map $\tau \rightarrow \ell$ that sends S to 1.

Proposition 4.1.5. *Let A be an algebra in Alg_ℓ^* . Then $A[t, t^{-1}]$ and $A \oplus \Sigma A$ are kk^h -equivalent.*

Proof. Consider the split extension

$$0 \rightarrow \sigma A \rightarrow A[t, t^{-1}] \rightarrow A \rightarrow 0;$$

therefore, from Remark 3.2.4, it follows that $A[t, t^{-1}]$ is kk^h equivalent to $A \oplus \sigma A$. We will show that σA is kk^h equivalent to ΣA . Since the coefficient ring A does not matter in the following proof, we omit it from notation. The proof follows like [CT07, Theorem 7.3.1 and Lemma 7.3.2].

Let $f : \tau \rightarrow A[t, t^{-1}]$ be the $*$ -morphism defined by $S \mapsto t$. This morphism restricts to $f| : \tau_0 \rightarrow \sigma$. On the other hand there is also a natural $*$ -morphism $g : \tau \rightarrow \Gamma$ sending S to the matrix

$$S \mapsto \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

It is easy to see that g is injective; thus, τ identifies with a $*$ -subalgebra of Γ . In this identification, the kernel of $f|$ is mapped to M_∞ . This gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_\infty & \longrightarrow & \tau_0 & \longrightarrow & \sigma & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_\infty & \longrightarrow & \Gamma & \longrightarrow & \Sigma & \longrightarrow & 0. \end{array}$$

If we show that τ_0 is kk^h -equivalent to 0, since we know that Γ is kk^h -equivalent to 0 by Corollary 3.2.14, we can use the five lemma and conclude that σ is kk^h -equivalent to Σ . For this we will construct a $*$ -homotopy from τ_0 to $M_\infty\tau[t]$ that when evaluated at $t = 0$ is the natural inclusion and is null when evaluated at $t = 1$.

First we define several $*$ -morphisms $\psi, \varphi_1, \varphi_2, \varphi_3 : \tau \rightarrow \tau \otimes \tau$ which are given by defining them on the generator S as

$$\begin{aligned} \psi(S) &= S^2S^* \otimes 1 & (4.1.6) \\ \varphi_1(S) &= S^2S^* \otimes 1 + (1 - SS^*) \otimes S \\ \varphi_2(S) &= S \otimes 1 \\ \varphi_3(S) &= S^2S^* \otimes 1 + (1 - SS^*) \otimes 1 \end{aligned}$$

All of these morphisms agree modulo the ideal $M_\infty\tau$. Identify τ with its image in Γ and define elements $u_t, v_t \in (\tau \otimes \tau)[t] \subseteq \Gamma\tau[t]$ by

$$u_t = \begin{pmatrix} 1 - SS^*t^2 & (t^3 - 2t)S & 0 & 0 & \cdots \\ tS^* & 1 - t^2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$v_t = \begin{pmatrix} 1 - t^2 & (t^3 - 2t) & 0 & 0 & \cdots \\ t & 1 - t^2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

It is readily checked that u_0, u_1, v_0 and v_1 are unitary matrices and that $1 - u_t$ and $1 - v_t$ belong in the ideal $M_\infty\tau[t]$. Write $U_t = c(u_t, u_t^{-1})$ and $V_t = c(v_t, v_t^{-1})$ as in Lemma 1.2.3. Define $\Phi_1, \Phi_2 : \tau \rightarrow M_\pm(\tau \otimes \tau)[t]$ as

$$\begin{aligned} \Phi_1(S) &= U_t i_+(S \otimes 1) \\ \Phi_2(S) &= V_t i_+(S \otimes 1). \end{aligned}$$

The following identities are then satisfied:

$$\begin{aligned} \text{ev}_0 \circ \Phi_1 &= \text{ev}_1 \circ \Phi_2 = i_+\varphi_2, \\ \text{ev}_1 \circ \Phi_1 &= i_+\varphi_1 \text{ and} \\ \text{ev}_1 \circ \Phi_2 &= i_+\varphi_3 \end{aligned}$$

Thus, restricting to τ_0 there are $*$ -quasi-homomorphisms

$$(\Phi_1, i_+\psi), (\Phi_2, i_+\psi) : \tau_0 \rightrightarrows M_{\pm}\tau \otimes \tau[t] \supseteq M_{\pm}M_{\infty}\tau[t].$$

Using Proposition 1.5.3 the $*$ -quasi-homomorphisms $(i_+\varphi_1, i_+\psi)$ and $(i_+\varphi_3, i_+\psi)$ induce the same morphisms in kk^h . Therefore, using hermitian stability the $*$ -quasi-homomorphisms (φ_1, ψ) and (φ_3, ψ) induce the same morphisms in kk^h .

Finally, since φ_1 is the orthogonal sum of ψ and the inclusion $\tau_0 \rightarrow M_{\infty}\tau$ and φ_3 agrees with ψ when restricted to τ_0 , using Proposition 1.5.3 this means that (φ_1, ψ) induces the same morphism as the inclusion $\tau_0 \rightarrow M_{\infty}\tau$ in kk^h and (φ_3, ψ) induces the null morphism. Thus the inclusion $\tau_0 \rightarrow M_{\infty}\tau$ is null on kk^h . By M_{∞} -stability this then implies that the inclusion $\tau_0 \rightarrow \tau$ is null, which implies that τ_0 is kk^h equivalent to 0 since the following extension is split:

$$0 \rightarrow \tau_0 \rightarrow \tau \rightarrow A \rightarrow 0 \quad \square$$

Pimsner-Voiculescu sequence

In topological K -theory, the Pimsner-Voiculescu sequence relates the K -theory groups of a crossed product $A \rtimes \mathbb{Z}$ with those of A . Here we present the algebraic analogue of this sequence in our setting.

Given a $*$ -automorphism $\sigma : A \rightarrow A$ we define the crossed product $A \rtimes_{\sigma} \mathbb{Z}$ as the ℓ -module $A[t, t^{-1}]$ but with multiplication given by the relation

$$tat^{-1} = \sigma(a)$$

and involution $(at)^* = t^{-1}\sigma(a)^*$.

Consider the $*$ -subalgebra τ_{σ} of $\tau \otimes_{\ell} (A \rtimes_{\sigma} \mathbb{Z})$ generated by $1 \otimes A$ and $S \otimes t$. This gives a semi-split extension

$$0 \rightarrow M_{\infty}A \rightarrow \tau_{\sigma} \rightarrow A \rtimes_{\sigma} \mathbb{Z} \rightarrow 0. \quad (4.1.7)$$

Proposition 4.1.8. *Let A be an algebra in Alg_{ℓ}^* . Then the sequence (4.1.7) induces the distinguished triangle in kk^h*

$$\Omega A \rightarrow A \xrightarrow{\text{id} - j^h(\sigma^{-1})} A \rightarrow A \rtimes_{\sigma} \mathbb{Z}.$$

Proof. Write $\kappa : A \rightarrow \tau_{\sigma}$ for the canonical inclusion. The same argument (with the obvious modifications) as in [Cun05, Propositions 14.1 and 14.2] shows that there is a commutative diagram in kk^h

$$\begin{array}{ccccccc} \Omega A \rtimes_{\sigma} \mathbb{Z} & \longrightarrow & A & \xrightarrow{\text{id} - j^h(\sigma^{-1})} & A & \longrightarrow & A \rtimes_{\sigma} \mathbb{Z} \\ \parallel & & \downarrow i & & \downarrow \kappa & & \parallel \\ \Omega A \rtimes_{\sigma} \mathbb{Z} & \longrightarrow & M_{\infty}A & \longrightarrow & \tau_{\sigma} & \longrightarrow & A \rtimes_{\sigma} \mathbb{Z}, \end{array}$$

so the statement of the proposition follows. \square

Cohn algebra of a graph

Let E be a directed graph, that is, a quadruple (E^0, E^1, r, s) where E^0 is the set of vertices of the graph and E^1 is the set of edges, $r, s : E^1 \rightarrow E^0$ are the source and range of the edges. A path in E is a sequence of edges $e_1 e_2 \cdots e_n$ where $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$; in this case we call n the length of the path. We define vertices to be paths of length 0. We define $\mathcal{P}(E)$ as the set of finite paths in E ; the range and source functions extend to $r, s : \mathcal{P}(E) \rightarrow E^0$ in the obvious way.

The *Cohn path algebra* $C(E)$ of a graph E is the $*$ -algebra generated by E^0 and E^1 subject to the relations

$$\begin{aligned} v \cdot w &= \delta_{v,w} v \\ v^* &= v \\ s(e) \cdot e &= e = r(e) \cdot e \\ e^* f &= \delta_{e,f} r(e) \end{aligned}$$

for v, w in E^0 and $e, f \in E^1$.

There is a natural morphism $\varphi : \ell(E^0) \rightarrow C(E)$ sending ev_v to v . For a vertex $v \in E^0$ such that $s^{-1}(v)$ is a finite set, we define

$$C(E) \ni m_v = \begin{cases} \sum_{e \in s^{-1}(v)} ee^* & \text{if } s^{-1}(v) \neq \emptyset \\ 0 & \text{if } s^{-1}(v) = \emptyset. \end{cases}$$

The elements m_v satisfy the identities

$$\begin{aligned} m_v &= m_v^*, \quad m_v^2 = m_v, \quad m_v w = \delta_{v,w} m_v, \\ m_v e &= \delta_{v,s(e)} e \quad (w \in E^0, e \in E^1). \end{aligned} \tag{4.1.9}$$

Write $C^m(E)$ for the algebra obtained from $C(E)$ by formally adjoining an element m_v for each vertex in E such that $s^{-1}(v)$ is infinite, subject to the identities (4.1). Let $q_v = v - m_v \in C^m(E)$ and write

$$\mathcal{K}(E) = \langle q_v | v \in E^0, s^{-1}(v) \text{ is finite non empty} \rangle \subseteq \widehat{\mathcal{K}}(E) = \langle q_v | v \in E^0 \rangle$$

for the corresponding ideals in $C^m(E)$.

Write $\widehat{i} : \ell(E^0) \rightarrow \widehat{\mathcal{K}}(E)$ for the $*$ -morphism that maps ev_v to q_v and let $\xi : C(E) \rightarrow C^m(E)$ be determined by

$$\xi(v) = m_v; \quad \xi(e) = em_{r(e)}.$$

The same argument as in [CM18, Remark 4.9] shows that \widehat{i} is a kk^h -equivalence. On other hand, the canonical inclusion $i : C(E) \rightarrow C^m(E)$ and ξ determine a $*$ -quasi-homomorphism $(i, \xi) : C(E) \rightrightarrows C^m(E) \supseteq \widehat{\mathcal{K}}(E)$. It is straightforward to see that $i\varphi = \xi\varphi + \widehat{i}$, therefore, using Proposition 1.5.3, we get

$$j^h(i, \xi)j^h(\varphi) = j^h(i\varphi, \xi\varphi) = j^h(\xi\varphi, \xi\varphi) + j^h(\widehat{i}, 0) = j^h(\widehat{i})$$

Hence, $j^h(\varphi)$ has a left inverse $j^h(\widehat{i})^{-1}j^h(i, \xi)$. We will show that $j^h(\varphi)$ is right invertible and therefore an isomorphism.

Consider $M_{\mathcal{P}(E)}$, the matrix ring indexed on the set $\mathcal{P}(E)$ and write $\epsilon_{\alpha, \beta}$ for its units. Define $\widehat{i}_\tau : C(E) \rightarrow M_{\mathcal{P}(E)}C(E)$ given on generators by

$$\widehat{i}_\tau(v) = \epsilon_{v,v} \otimes v, \quad \widehat{i}_\tau(e) = \epsilon_{s(e), r(e)} \otimes e$$

for $v \in E^0$ and $e \in E^1$. Also define $\widehat{\varphi} : \widehat{\mathcal{K}}(E) \rightarrow M_{\mathcal{P}(E)}C(E)$ by

$$\widehat{\varphi}(\alpha q_v \beta^*) = \epsilon_{\alpha, \beta} \otimes v.$$

There is a commutative diagram

$$\begin{array}{ccc} \ell(E^0) & \xrightarrow{\widehat{i}} & \widehat{\mathcal{K}}(E) \\ \downarrow \varphi & & \downarrow \widehat{\varphi} \\ C(E) & \xrightarrow{\widehat{i}_\tau} & M_{\mathcal{P}(E)}C(E). \end{array}$$

Lemma 4.1.10. *Let $\alpha \in \mathcal{P}(E)$ be a path and $i_\alpha : C(E) \rightarrow M_{\mathcal{P}(E)}C(E)$ be the inclusion in the α -diagonal coordinate $(e_{\alpha, \alpha})$. Then \widehat{i}_τ and i_α induce the same isomorphism in kk^h .*

Proof. Using Lemma 1.6.8, the class of i_α does not depend on α since i_α and i_β are conjugates. So we assume $\alpha = w \in E^0$. For each $v \in E^0$, $v \neq w$ write

$$\begin{aligned} a_v &= (1 - t^2)\epsilon_{w,w} + (t^3 - 2t)\epsilon_{w,v} + t\epsilon_{v,w} + (1 - t^2)\epsilon_{v,v} \\ b_v &= (1 - t^2)\epsilon_{w,w} + (2t - t^3)\epsilon_{w,v} - t\epsilon_{v,w} + (1 - t^2)\epsilon_{v,v} \\ a_w &= b_w = \epsilon_{w,w}. \end{aligned}$$

Define $C_v = c(a_v, b_v)$ as in Lemma 1.2.3. Then we have a $*$ -homotopy $H : C(E) \rightarrow M_{\pm}M_{\mathcal{P}(E)}C(E)[t]$ given by

$$\begin{aligned} H(v) &= C_v i_+(\epsilon_{v,v} \otimes v) C_v \\ H(e) &= C_{s(e)} i_+(\epsilon_{s(e), r(e)} \otimes e) C_{r(e)}^* \end{aligned}$$

which satisfies $\text{ev}_0 H = i_+ \widehat{i}_\tau$ and $\text{ev}_1 H = i_+ i_w$. Using hermitian stability we conclude that \widehat{i}_τ and i_w are the same in kk^h . \square

Write $\mathfrak{A} \subseteq M_{\mathcal{P}(E)}C(E)$ for the ℓ -submodule generated by

$$\mathfrak{A} = \text{span}\{e_{\gamma, \delta} \otimes \alpha \beta^* \in M_{\mathcal{P}(E)}C(E) : s(\alpha) = r(\gamma), s(\beta) = r(\delta), r(\alpha) = r(\beta)\}.$$

It is readily checked that \mathfrak{A} is a $*$ -subalgebra of $M_{\mathcal{P}(E)}C(E)$, and $\text{Im } \widehat{i}_\tau, \text{Im } \widehat{\varphi} \subseteq \mathfrak{A}$. In particular, \widehat{i}_τ restricted to \mathfrak{A} induces a monomorphism in kk^h .

Let $\Gamma_{\mathcal{P}(E)}$ be the cone algebra indexed by $\mathcal{P}(E)$. There is a $*$ -morphism $\rho : C^m(E) \rightarrow \Gamma_{\mathcal{P}(E)}$ given by

$$\begin{aligned}\rho(v) &= \sum_{s(\alpha)=v} \epsilon_{\alpha,\alpha} \\ \rho(e) &= \sum_{r(\alpha)=r(e)} \epsilon_{e\alpha,\alpha} \\ \rho(m_w) &= \sum_{\substack{r(\alpha)=w \\ \text{length } \alpha \geq 1}} \epsilon_{\alpha,\alpha}.\end{aligned}$$

Consider the $*$ -morphism $\rho' = \rho \otimes 1 : C^m(E) \rightarrow \Gamma_{\mathcal{P}(E)} \widetilde{C^m(E)}$. Then \mathfrak{A} is closed by multiplication by elements on the image of ρ' on both sides, so we can form the semi-direct product $C^m(E) \rtimes \mathfrak{A}$. Define the algebra D as the quotient of $C^m(E) \rtimes \mathfrak{A}$ by the $*$ -ideal

$$\langle \alpha q_v \beta^*, -\epsilon_{\alpha,\beta} \otimes v : v = r(\alpha) = r(\beta) \rangle.$$

It is shown on [CM18, Lemma 4.19] that \mathfrak{A} maps injectively to D , meaning it is isomorphic to an ideal inside D . We then have a commutative diagram

$$\begin{array}{ccccc} \ell(E^0) & \xrightarrow{\widehat{i}} & \widehat{\mathcal{K}}(E) & \longrightarrow & C^m(E) \\ \downarrow \varphi & & \downarrow \widehat{\varphi} & & \downarrow \Xi \\ C(E) & \xrightarrow{\widehat{i}_\tau} & \mathfrak{A} & \longrightarrow & D, \end{array} \quad (4.1.11)$$

where Ξ is given by the composition of the inclusion $C^m(E) \rightarrow C^m(E) \rtimes \mathfrak{A}$ and the projection $C^m(E) \rtimes \mathfrak{A} \rightarrow D$. Define $\psi_0 = \Xi$, $\psi_1 = \Xi \xi$. It is easy to check that ψ_1 is orthogonal to \widehat{i}_τ so we can define $\psi_{1/2} = \psi_1 + \widehat{i}_\tau$. These $*$ -morphisms define $*$ -quasi-homomorphisms

$$(\psi_0, \psi_1), (\psi_0, \psi_{1/2}), (\psi_{1/2}, \psi_1) : C(E) \rightrightarrows D \triangleright \mathfrak{A}$$

Lemma 4.1.12. *The $*$ -quasi-homomorphism $(\psi_0, \psi_{1/2})$ induces the zero map in kk^h .*

Proof. For each $e \in E^1$ consider the matrices in $\Gamma_{\mathcal{P}(E)} C(E)[t]$

$$\begin{aligned}u_t^e &= \epsilon_{s(e),s(e)}(1-t^2) \otimes ee^* + \epsilon_{e,s(e)} \otimes te^* \\ v_t^e &= \epsilon_{s(e),s(e)}(1-t^2) \otimes ee^* + \epsilon_{s(e),e} \otimes (2t-t^3)e.\end{aligned}$$

Observe that multiplying by u_t^e and v_t^e preserves \mathfrak{A} . Put $U_t^e = c((0, u_t^e), (0, v_t^e)) \in M_{\pm} D[t]$ and define a $*$ -homotopy $H : C(E) \rightarrow M_{\pm} D[t]$ determined by

$$\begin{aligned}H(v) &= i_+(v, 0) & (v \in E^0) \\ H(e) &= i_+(em_{r(e)}, 0) + U_t^e i_+(0, \epsilon_{s(e),r(e)} \otimes e) & (e \in E^1)\end{aligned}$$

Then H is a $*$ -homotopy between ψ_0 and $\psi_{1/2}$ and the $*$ -quasi-homomorphism $(H, i_+ \psi_{1/2})$ is a $*$ -homotopy between $(\psi_0, \psi_{1/2})$ and $(\psi_{1/2}, \psi_{1/2})$. Therefore, by Proposition 1.5.3 $j^h(\psi_0, \psi_{1/2})$ is the zero morphism \square

Theorem 4.1.13. *The morphism $\varphi : \ell^{(E^0)} \rightarrow C(E)$ is a kk^h -equivalence.*

Proof. We have already checked that

$$j^h(\widehat{i})^{-1}j^h(i, \xi)j^h(\varphi) = j^h(\text{id}_{\ell^{(E^0)}}).$$

The commutative diagram (4.1.11) and the previous lemma show that

$$j^h(\widehat{\varphi})j^h(i, \xi) = j^h(\psi_0, \psi_1) = j^h(\psi_0, \psi_{1/2}) + j^h(\psi_{1/2}, \psi_1) = j^h(\psi_{1/2}, \psi_1) = j^h(\widehat{i}_\tau)$$

And on other hand

$$j^h(\widehat{\varphi})j^h(i, \xi) = j^h(\widehat{i}_\tau)j^h(\varphi)j^h(\widehat{i})^{-1}j^h(i, \xi)$$

hence

$$j^h(\widehat{i}_\tau) = j^h(\widehat{i}_\tau)j^h(\varphi)j^h(\widehat{i})^{-1}j^h(i, \xi),$$

and since $j^h(\widehat{i}_\tau)$ is a monomorphism, this shows that

$$j^h(\text{id}_{C(E)}) = j^h(\varphi)j^h(\widehat{i})^{-1}j^h(i, \xi)$$

as we wanted. □

4.2 Comparison of with KH^h

Theorem 4.2.1 ([cf. CT07, Theorem 8.2.1]). *The map from Example 3.2.34 gives an isomorphism*

$$KH_0^h(A) \cong kk^h(\ell, A).$$

Proof. Suppose first A unital, the general case follows from excision. Recall from Remark 2.1.6 the set of $*$ -quasi-homomorphisms $qq(\ell, A)$ and the surjective map

$$qq(\ell, A) \rightarrow K_0^h(A).$$

Using Example 1.5.2, for $(e_0, e_1) \in qq(\ell, A)$, this $*$ -quasi-homomorphism also induces a map $(e_i) : q\ell \rightarrow {}_1M_2^h M_\infty A$ which induces a map in kk^h ,

$$\begin{aligned} qq(\ell, A) &\rightarrow kk^h(q\ell, {}_1M_2^h M_\infty A) \cong kk^h(q\ell, A) \\ (e_0, e_1) &\mapsto [e_i] \end{aligned} \tag{4.2.2}$$

Using Lemma 1.6.8, the map (4.2.2) sends equivalent classes in K_0^h to the same morphism in kk^h , so the map then factors as

$$\begin{array}{ccc} qq(\ell, A) & \longrightarrow & kk^h(q\ell, A) \\ \downarrow & \nearrow & \\ K_0^h(A) & & \end{array}$$

Using Corollary 4.1.4 we have that $\pi_0 : q\ell \rightarrow \ell$ induces a kk^h -equivalence, therefore we can compose to get

$$K_0^h(A) \rightarrow kk^h(\ell, A).$$

Using excision for K_n^h for $n \leq 0$, this gives a map

$$\alpha : KH_0^h(A) = \underset{n}{\operatorname{colim}} K_0^h(\Sigma^n \Omega^n A) \rightarrow \underset{n}{\operatorname{colim}} kk^h(\ell, \Sigma^n \Omega^n A) = kk(\ell, A).$$

Write

$$\beta : kk^h(\ell, A) \rightarrow KH_0^h(A)$$

for the map in Example 3.2.34. We will show that α and β are inverses of each other.

Using the description of β given in Remark 3.2.23, for a self-adjoint idempotent $e \in {}_1M_2^h M_\infty A$, where A is unital, is immediate to see that $\beta\alpha(c_0[e]) = c_0[e]$. This implies that $\beta\alpha$ is the identity in $KH_0^h(A)$.

To complete the proof we will show that α is surjective. Let $\varphi : J^n(\ell) \rightarrow M_\infty M_\pm^{\otimes k} A^{\operatorname{sd}^r S^n}$ represent a class in $kk^h(\ell, A)$. Using the map $\ell \rightarrow \Sigma^n J^n(\ell)$ in kk^h consider the induced map on KH_0^h

$$KH_0^h(\ell) \rightarrow KH_0^h(\Sigma^n J^n(\ell))$$

and write $e \in KH_0^h(\Sigma^n J^n(\ell))$ for the image of the element $[1] \in KH_0^h(\ell)$. It follows from the definition of α , that $\alpha([1])$ equals id_ℓ in $kk^h(\ell, \ell)$. Let then $\kappa : q\ell \rightarrow {}_1M_2 M_\infty \Sigma^n J^n(\ell)$ be the associated map to a $*$ -quasi-homomorphism that induces e . Thus we have the follows equality in $kk^h(\ell, \Sigma^n J^n \ell)$

$$j^h(\kappa)j^h(\pi_0)^{-1} = \alpha(e). \quad (4.2.3)$$

In turn, this shows that $j^h(\kappa)j^h(\pi_0)^{-1}$ is the morphism that induces the kk^h equivalence $\ell \sim \Sigma^n J^n(\ell)$. On other hand, consider the commutative diagram in kk^h

$$\begin{array}{ccc} \Sigma^n J^n(\ell) & \xrightarrow{\Sigma^n \varphi} & \Sigma^n A^{\operatorname{sd}^r S^n} \\ \wr \uparrow & & \wr \uparrow \\ \ell & \xrightarrow{\varphi} & A \end{array}$$

where the right arrow is an isomorphism because of Corollary 3.2.8. It follows that $(\Sigma^n \varphi)_*(e) \in KH_0^h(\Sigma^n A^{\operatorname{sd}^r S^n}) \cong KH_0^h(A)$ is the same class as $j^h(\varphi)_*([1]) \in KH_0^h(A)$. Therefore, using (4.2.3) we have following equalities in $kk^h(\ell, A)$:

$$\alpha(\varphi_*([1])) = \alpha((\Sigma^n \varphi)_*(e)) = (\Sigma^n \varphi)\alpha(e) = (\Sigma^n \varphi)j^h(\kappa)j^h(\pi_0)^{-1} = \varphi.$$

This concludes the proof. \square

Chapter 5

Karoubi's Fundamental Theorem in kk^h

In this chapter we prove an analogous result to Theorem 2.3.3 in the category kk^h . For this, we develop some preliminary results about the induction and restriction functors in Section 5.1; we then define functors $U, V : Alg_\ell^* \rightarrow Alg_\ell^*$ in Section 5.2, which are similar to the functors U', V' described in Section 2.3 and which in kk^h give equivalent functors up to suspension/looping; we also show that functors U, V satisfy analogous properties to the ones discussed in Section 2.3. Finally in Section 5.3 we use the functors U, V and the properties that were discussed in Section 5.2 to conclude Theorem 5.3.1 and Theorem 5.3.7.

5.1 The functors ind , res and Λ

Recall the functors $\text{res} : kk^h \rightarrow kk$, and $\text{ind}, \text{ind}' : kk \rightarrow kk^h$ from Example 3.2.28.

Proposition 5.1.1. *The functors $\text{res} : kk^h \leftrightarrow kk : \text{ind}$ are both right and left adjoint to one another; in other words, for every $A \in Alg_\ell^*$ and $B \in Alg_\ell$ there are natural isomorphisms*

$$kk(\text{res}(A), B) \cong kk^h(A, \text{ind}(B)) \text{ and } kk^h(\text{ind}(B), A) \cong kk(B, \text{res}(A)).$$

Proof. Using Proposition 4.1.1, the functors $\text{ind}, \text{ind}' : kk \rightarrow kk^h$ are naturally equivalent, since one is right adjoint to res and the other is left adjoint, the result follows. \square

Remark 5.1.2. The unit and counit maps of the second adjunction in Proposition 5.1.1 are obtained from those of the adjunction between ind' and res using the projection $\pi : \text{ind}' \rightarrow \text{ind}$ and the diagonal map $\text{ind} \rightarrow M_2 \text{ind}'$ as in the proof of Proposition 4.1.1.

Let $\Lambda = \ell \oplus \ell$ equipped with involution

$$(\lambda, \mu)^* = (\mu^*, \lambda^*).$$

For $A \in \text{Alg}_\ell^*$ write ΛA for $\Lambda \otimes A$ and $\Lambda : \text{Alg}_\ell^* \rightarrow \text{Alg}_\ell^*$ for the associated functor.

Recall from Section 2.3 that for $A \in \text{Alg}_\ell^*$ we write $\widehat{A} = \text{ind}(\text{res}(A))$. Then $\widehat{A} \cong \Lambda A$ via the isomorphism

$$\begin{aligned} \Lambda A &\rightarrow \widehat{A} \\ (x, y) &\mapsto (x, y^*). \end{aligned}$$

Under this identification, the maps η_A of (2.3.2) and φ_A of (2.3.1) become the scalar extensions of the embeddings

$$\eta : \ell \rightarrow \Lambda \tag{5.1.3}$$

$$x \mapsto (x, x),$$

$$\phi : \Lambda \rightarrow {}_1M_2 \tag{5.1.4}$$

$$(x, y) \mapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Remark 5.1.5. The functor induced by tensoring with $\Lambda : kk^h \rightarrow kk^h$ is left and right adjoint to itself since $\Lambda \cong \text{ind res}(\cdot)$. Also, Proposition 5.1.1 shows that

$$kk^h(\cdot, \Lambda(\cdot)) \cong kk(\text{res}(\cdot), \text{res}(\cdot)).$$

In other words, Λ represents kk . In particular, we have

$${}_\varepsilon kk^h(\cdot, \Lambda(\cdot)) \cong kk^h(\cdot, \Lambda(\cdot))$$

for any unitary $\varepsilon \in \ell$. Moreover by Remark 1.1.26, if $R \in \text{Alg}_\ell^*$ is unital and $\varepsilon \in R$ is central unitary and $\Psi \in R$ is an invertible ε -hermitian element, then we have an $*$ -isomorphism

$$\text{ad}(1, \Psi^{-1}) : \Lambda R \rightarrow \Lambda R^\Psi.$$

In particular, we have $*$ -isomorphisms

$$\Lambda M_\pm \cong \Lambda({}_\varepsilon M_2) \cong \Lambda M_2.$$

Remark 5.1.6. Let $t : \Lambda \rightarrow \Lambda$ defined as $t(x, y) = (y, x)$. Then t is a $*$ -automorphism, with $t^2 = \text{id}_\Lambda$; moreover using Remark 5.1.5 one checks that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{(\text{id}_{{}_1M_2} \otimes \eta)\phi} & {}_1M_2\Lambda \\ & \searrow & \downarrow \wr \\ & \begin{pmatrix} \text{id} & 0 \\ 0 & t \end{pmatrix} & M_2\Lambda. \end{array}$$

Thus, using Lemma 1.6.9 we get

$$j^h(i_2)^{-1}j^h(\eta)j^h(\phi) = j^h(\text{id}_\Lambda) + j^h(t).$$

5.2 The functors U and V

Consider the path algebras (Example 1.4.5) of the maps (5.1.4) and (5.1.3),

$$U = P_\phi \text{ and } V = P_\eta.$$

For $A \in Alg_\ell^*$, write $UA = U \otimes A$ and $VA = V \otimes A$; these are, respectively, the path algebras of $\phi \otimes \text{id}_A : \Lambda A \rightarrow {}_1M_2A$ and $\eta \otimes \text{id}_A : A \rightarrow \Lambda A$. Because U and V are flat ℓ -modules, they define functors $U, V : kk^h \rightarrow kk^h$ by Example 3.2.29.

Remark 5.2.1. Recall the functors U', V' from Section 2.3. Using Remark 3.2.24 and the isomorphism $\widehat{A} \cong \Lambda A$, it follows that there are kk^h -equivalences

$$\begin{aligned} U &\sim \Omega U' \ell \\ V &\sim \Omega V' \ell. \end{aligned}$$

In Lemmas 5.2.2 and 5.2.6 we recast the equivalences of (2.3.4) into the framework of kk^h .

Lemma 5.2.2. *There are kk^h -equivalences*

$$\begin{aligned} U\Lambda &\sim \Lambda \text{ and} \\ V\Lambda &\sim \Omega\Lambda. \end{aligned}$$

Proof. Let us prove the first equivalence. To ease the notation we omit the functor j^h . Let

$$\Omega_1 M_2 \Lambda \rightarrow U\Lambda \rightarrow \Lambda^2 \xrightarrow{\phi \otimes \text{id}_\Lambda} {}_1M_2\Lambda$$

be the triangle in kk^h induced by the extension which defines $U\Lambda$. We have an isomorphism

$$\begin{aligned} \tau : \Lambda^2 &\cong \Lambda \oplus \Lambda \\ (x_1, x_2) \otimes (x_3, x_4) &\mapsto (x_1x_3, x_2x_4, x_1x_4, x_2x_3). \end{aligned} \tag{5.2.3}$$

Put $\lambda_1 = (0, 1)$, $\lambda_2 = (1, 0)$ and $\iota_i : \Lambda \rightarrow {}_1M_2\Lambda$ as in Lemma 3.1.11. Let $j_i : \Lambda \rightarrow \Lambda \oplus \Lambda$ ($i = 1, 2$) be the inclusions in each coordinate. Observe that

$$((\phi \otimes \text{id}_\Lambda) \circ \tau^{-1} \circ j_1)(x, y) = \begin{pmatrix} (x, 0) & (0, 0) \\ (0, 0) & (0, y) \end{pmatrix}.$$

The matrix

$$u = \begin{pmatrix} (1, -1) & (1, 1) \\ (0, 0) & (-1, 1) \end{pmatrix} \in {}_1M_2(\text{ind}(\tilde{B})) \tag{5.2.4}$$

is unitary and satisfies

$$\text{ad}(u) \circ \iota_1 = (\phi \otimes \text{id}_\Lambda) \circ \tau^{-1} \circ j_1 : \Lambda \rightarrow {}_1M_2\Lambda.$$

So by Lemma 1.6.8, the following diagram commutes in kk^h

$$\begin{array}{ccc} \Lambda & \xrightarrow{j_1} & \Lambda \oplus \Lambda \\ & \searrow \iota_1 \sim & \downarrow (\phi \otimes \text{id}_\Lambda) \circ \tau^{-1} \\ & & {}_1M_2\Lambda. \end{array}$$

Similarly, the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{j_2} & \Lambda \oplus \Lambda \\ & \searrow \iota_2 \sim & \downarrow (\phi \otimes \text{id}_\Lambda) \circ \tau^{-1} \\ & & {}_1M_2\Lambda \end{array}$$

commutes in kk^h . Let $pr_i : \Lambda \oplus \Lambda \rightarrow \Lambda$ ($i = 1, 2$) be the projections on each coordinate. By Lemma 3.1.11 we have $j^h(\iota_1) = j^h(\iota_2)$; thus, using the previous diagrams, the following solid arrow diagram commutes in kk^h :

$$\begin{array}{ccccc} U\Lambda & \longrightarrow & \Lambda^2 & \xrightarrow{\varphi \otimes \text{id}_\Lambda} & {}_1M_2\Lambda \\ \uparrow \text{---} & & \uparrow \tau^{-1} \wr & & \uparrow \iota_1 = \iota_2 \\ \Lambda & \xrightarrow{j_1 - j_2} & \Lambda \oplus \Lambda & \xrightarrow{\pi_1 + \pi_2} & \Lambda. \end{array}$$

Since the lower row is split, it completes to a triangle by Remark 3.2.4. Then, because the middle and right vertical arrows are isomorphisms in kk^h , we get that the dashed map is an isomorphism in kk^h .

Next we prove the second isomorphism of the statement. Let

$$\Omega\Lambda^2 \rightarrow V\Lambda \rightarrow \Lambda \xrightarrow{\eta \otimes \text{id}_\Lambda} \Lambda^2$$

be the triangle in kk^h induced by the extension defining $V\Lambda$. Let t be as in (5.1.6); one checks that the following square commutes

$$\begin{array}{ccc} \Lambda & \xrightarrow{\eta \otimes \text{id}_\Lambda} & \Lambda^2 \\ \parallel & & \downarrow \tau \\ \Lambda & \xrightarrow{j_1 + j_2 t} & \Lambda \oplus \Lambda. \end{array} \quad (5.2.5)$$

The map $j_1 + j_2 t$ completes to a split distinguished triangle in kk^h

$$\Omega\Lambda \rightarrow \Lambda \xrightarrow{j_1 + j_2 t} \Lambda \oplus \Lambda \xrightarrow{\pi_1 - t\pi_2} \Lambda.$$

Rotating the split triangle above we get the triangle

$$\Omega(\Lambda \oplus \Lambda) \rightarrow \Omega\Lambda \xrightarrow{0} \Lambda \xrightarrow{j_1 + j_2 t} \Lambda \oplus \Lambda.$$

Finally, (5.2.5) extends to a commutative diagram in kk^h :

$$\begin{array}{ccccc} V\Lambda & \longrightarrow & \Lambda & \xrightarrow{\eta \otimes \text{id}_\Lambda} & \Lambda^2 \\ \downarrow \text{---} & & \parallel & & \downarrow \tau \\ \Omega\Lambda & \xrightarrow{0} & \Lambda & \xrightarrow{j_1 + j_2 t} & \Lambda. \end{array}$$

It follows that the dashed map is an isomorphism. \square

Lemma 5.2.6. *There is a kk^h -equivalence*

$$\Sigma VU \sim \ell.$$

In particular, $VU \sim \Omega$.

Proof. As before, we omit j^h from the notation. In view of Lemma 3.1.11, it suffices to show that ΣVU is kk^h -equivalent to ${}_1M_2$. Let

$$\Omega\Lambda U \rightarrow VU \rightarrow U \xrightarrow{\eta \otimes \text{id}_U} \Lambda U$$

be the triangle in kk^h induced by the extension that defines VU . The kk^h isomorphism between $\Lambda U = U\Lambda$ and Λ established in Lemma 5.2.2 is induced by mapping Λ^2 to $\Lambda \oplus \Lambda$ and then retracting onto the first coordinate. Using this fact we get that there is a map of triangles in kk^h

$$\begin{array}{ccccccc} U & \xrightarrow{\eta \otimes \text{id}_U} & \Lambda U & \longrightarrow & \Sigma VU & \longrightarrow & \Sigma U \\ \parallel & & \wr \downarrow & & \downarrow & & \parallel \\ U & \longrightarrow & \Lambda & \xrightarrow{\phi} & {}_1M_2 & \longrightarrow & \Sigma U. \end{array}$$

It follows that the dashed kk^h -map is an isomorphism. \square

Remark 5.2.7. By Example 3.2.29, the isomorphisms of Lemmas 5.2.2 and 5.2.6 induce kk^h -equivalences $U\Lambda A \sim \Lambda A$, $V\Lambda A \sim \Omega\Lambda A$ and $VUA \sim \Omega A$ for every $A \in \text{Alg}_\ell^*$.

5.3 Bivariant version of Karoubi's Fundamental Theorem

Recall from Corollary 2.3.6, the element $\theta \in KH_2^h({}_{-1}M_2(U')^2\ell)$. Using Remark 5.2.1 and Theorem 4.2.1, we get an element $\theta \in kk^h(\ell, {}_{-1}M_2U^2)$. Also, recall the product induced by the tensor product from Proposition 3.2.30.

Theorem 5.3.1. *For all $A \in \text{Alg}_\ell^*$, the product with θ induces a natural isomorphism*

$$\theta_A := \theta \otimes j^h(\text{id}_A) : j^h(A) \cong j^h({}_{-1}M_2U^2A).$$

Proof. By Example 3.2.29, it suffices to show that $\theta = \theta_\ell$ is an isomorphism. Equivalently, we need to see that

$$\begin{aligned} kk^h(\ell, \theta)_* : kk^h(\ell, \ell) &\rightarrow kk^h(\ell, {}_{-1}M_2U^2) \text{ and} \\ kk^h({}_{-1}M_2U^2, \theta)_* : kk^h({}_{-1}M_2U^2, \ell) &\rightarrow kk^h({}_{-1}M_2U^2, {}_{-1}M_2U^2) \end{aligned}$$

are isomorphisms.

Taking into account hermitian stability and using Lemma 5.2.6, we see that $kk^h(-_1M_2U^2, \theta)_*$ is an isomorphism if and only if

$$kk^h(\ell, \theta_{-1M_2(\Sigma V)^2})_* : kk^h(\ell, -_1M_2(\Sigma V)^2) \rightarrow kk(\ell, -_1M_2(\Sigma VU)^2)$$

is an isomorphism. Hence the theorem will follow if we prove that $(\theta_A)_* := kk^h(\ell, \theta_A)$ is an isomorphism for all A .

By Proposition 3.2.35 and the isomorphism of Theorem 4.2.1, the map $(\theta_A)_*$ corresponds to the cup-product with θ , which by Corollary 2.3.6 is an isomorphism. \square

Corollary 5.3.2. *Let $\varepsilon \in \ell$ be unitary. For every $A \in Alg_\ell^*$, there is a kk^h -equivalence*

$${}_\varepsilon M_2VA \sim {}_{-\varepsilon} M_2U\Omega A.$$

Proof. It is immediate from Theorem 5.3.1, Lemma 5.2.6 and Remark 5.2.7 that $VA \sim {}_{-1}M_2U\Omega A$. The corollary follows from this applied to ${}_\varepsilon M_2A$ using the isomorphism

$${}_{-1}M_2({}_\varepsilon M_2) \cong M_\pm({}_{-\varepsilon} M_2)$$

and hermitian stability. \square

Lemma 5.3.3. *Consider the kk^h -equivalences $U\Lambda \sim \Lambda$ of Lemma 5.2.2 and $M_2\Lambda \cong {}_{-1}M_2\Lambda$ of Remark 5.1.5. Then the following diagram commutes in kk^h :*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\theta_\Lambda} & {}_{-1}M_2U^2\Lambda \\ \downarrow i_2 & & \downarrow \wr \\ M_2\Lambda & \xrightarrow{\sim} & {}_{-1}M_2\Lambda \end{array}$$

Proof. By part i) of Theorem 2.3.3, we have a commutative diagram in kk^h , where as usual we have omitted j^h ,

$$\begin{array}{ccccc} \ell & \xrightarrow{\theta} & {}_{-1}M_2U^2 & \longrightarrow & {}_{-1}M_2\Lambda U \\ \downarrow i_{2\eta} & & & & \downarrow \\ M_2\Lambda & \xrightarrow{\sim} & & \longrightarrow & {}_{-1}M_2\Lambda. \end{array} \quad (5.3.4)$$

Let $p = pr_1 \circ \tau : \Lambda^2 \rightarrow \Lambda$; we have

$$p((x_1, x_2) \otimes (x_3, x_4)) = (x_1x_3, x_2x_4).$$

Tensoring (5.3.4) with Λ and composing the resulting vertical maps with those induced by p , we get another commutative diagram

$$\begin{array}{ccccc} \Lambda & \xrightarrow{\theta_\Lambda} & {}_{-1}M_2U^2\Lambda & \longrightarrow & {}_{-1}M_2\Lambda U\Lambda \\ \downarrow i_2 & & & & \downarrow \\ M_2\Lambda & \xrightarrow{\sim} & & \longrightarrow & {}_{-1}M_2\Lambda. \end{array} \quad (5.3.5)$$

Using the fact that the kk^h -equivalence $U\Lambda \sim \Lambda$ is induced by first mapping to Λ^2 and then applying p , we obtain a commutative diagram in kk^h

$$\begin{array}{ccc} U^2\Lambda & \longrightarrow & \Lambda U\Lambda \\ \downarrow & \swarrow & \\ \Lambda & & \end{array}$$

Tensoring with ${}_{-1}M_2$ we obtain that the composite ${}_{-1}M_2U^2\Lambda \rightarrow {}_{-1}M_2\Lambda$ in diagram (5.3.5) is the map in the diagram of the proposition, finishing the proof. \square

The bivariant 12-term exact sequence

Definition 5.3.6 (cf. Definition 2.3.7). Let $A, B \in \text{Alg}_\ell^*$, $\varepsilon \in \ell$ unitary, ${}_\varepsilon kk^h(A, B)$ as in Definition 3.2.31 and t as in (5.1.6). Let $\eta : \ell \rightarrow \Lambda$ and $\varphi : \Lambda \rightarrow {}_1M_2$ be as in (5.1.3) and (5.1.4). Put $\bar{\varphi} = j^h(\iota_1)^{-1} \circ j^h(\varphi)$. Set

$$\begin{aligned} {}_\varepsilon W(A, B) &:= \text{coker}({}_\varepsilon kk^h(A, \Lambda B) \xrightarrow{\bar{\varphi}_*} {}_\varepsilon kk^h(A, B)) \\ {}_\varepsilon W'(A, B) &:= \text{ker}({}_\varepsilon kk^h(A, B) \xrightarrow{\eta_*} {}_\varepsilon kk^h(A, \Lambda B)) \\ k(A, B) &:= \{x \in kk^h(A, \Lambda B) : x = t_*x\} / \{x = y + t_*y \text{ for some } y\} \\ k'(A, B) &:= \{x \in kk^h(A, \Lambda B) : x = -t_*x\} / \{x = y - t_*y \text{ for some } y\} \end{aligned}$$

If $\varepsilon = 1$ we omit it from the notation. Note that k and k' do not need the ε prescript due to the isomorphism in Remark 5.1.5.

Theorem 5.3.7 ([cf. Kar80, Théorème 4.3]). *There is an exact sequence*

$$\begin{array}{ccccccccc} k(A, \Omega B) & \longrightarrow & {}_{-1}W(A, \Omega^2 B) & \longrightarrow & W'(A, B) & \longrightarrow & k'(A, \Omega B) & \longrightarrow & {}_{-1}W'(A, \Omega B) & \longrightarrow & {}_{-1}W(A, \Omega B) \\ \uparrow & & & & & & & & & & \downarrow \\ W(A, \Omega B) & \longleftarrow & W'(A, \Omega B) & \longleftarrow & k'(A, \Omega B) & \longleftarrow & {}_{-1}W'(A, B) & \longleftarrow & W(A, \Omega^2 B) & \longleftarrow & k(A, \Omega B) \end{array}$$

Proof. As above, we omit j^h in our notation. Write ν for the map obtained upon tensoring the canonical map $U \rightarrow \Lambda$ with $\Omega_{-1}M_2$. Consider the following distinguished triangles in kk^h

$$\begin{array}{ccccccc} \Omega\Lambda & \xrightarrow{\partial} & V & \longrightarrow & \ell & \xrightarrow{\eta} & \Lambda \\ & & \downarrow \theta & & & & \\ \Omega^2{}_{-1}M_2 & \xrightarrow{\delta} & \Omega_{-1}M_2U & \xrightarrow{\nu} & \Omega_{-1}M_2\Lambda & \xrightarrow{\Omega_{-1}M_2\phi} & \Omega_{-1}M_2. \end{array}$$

Recall $\tau : \Lambda^2 \cong \Lambda \oplus \Lambda$ from (5.2.3) and let $\tilde{\tau} : \Omega_{-1}M_2\Lambda^2 \rightarrow \Omega(\Lambda \oplus \Lambda)$ be the composite in kk^h of the isomorphism Remark 5.1.5, the inverse of the corner inclusion, and

$\Omega\tau$. Using Lemma 5.3.3 we get the following commutative diagram in kk^h :

$$\begin{array}{ccccccc}
\Omega\Lambda & \xrightarrow{\partial} & V & \xrightarrow{\theta} & \Omega_{-1}M_2U & \xrightarrow{\nu} & \Omega_{-1}M_2\Lambda \\
\downarrow \Omega\Lambda\eta & & \downarrow V\eta & & \downarrow \Omega_{-1}M_2U\eta & & \downarrow \Omega_{(-1}M_2\Lambda)\eta \\
\Omega\Lambda^2 & \xrightarrow{\partial} & V\Lambda & \xrightarrow{\theta} & \Omega_{-1}M_2U\Lambda & \xrightarrow{\nu} & \Omega_{-1}M_2\Lambda^2 \\
\downarrow \Omega\tau & & \downarrow \wr & & \downarrow \wr & & \downarrow \tilde{\tau} \\
\Omega(\Lambda \oplus \Lambda) & \xrightarrow{\Omega(\pi_1 - t\pi_2)} & \Omega\Lambda & \xlongequal{\quad} & \Omega\Lambda & \xrightarrow{\Omega(j_1 - j_2)} & \Omega(\Lambda \oplus \Lambda).
\end{array}$$

A direct computation shows that $\tau \circ (\Lambda\eta) : \Lambda \rightarrow \Lambda \oplus \Lambda$ is the diagonal map. Hence from the diagram we get following equality in $kk^h(\Omega\Lambda, \Omega(\Lambda \oplus \Lambda))$

$$\tilde{\tau}(\Omega_{-1}M_2\eta)\nu\theta\partial = \Omega((j_1 - j_2)(\pi_1 - t\pi_2)(j_1 + j_2)). \quad (5.3.8)$$

Similarly, for h_{-1} as in Example 1.1.18 and i_2 the upper left-hand corner inclusion, we have in $kk^h(\Omega_{-1}M_2\Lambda, \Omega(\Lambda \oplus \Lambda))$

$$\tilde{\tau}(\Omega_{-1}M_2\eta) = \Omega(j_1 + j_2)(i_2)^{-1} \text{ad}(1, h_{-1}^{-1}).$$

Therefore, composing both sides of the equality (5.3.8) on the left with the projection onto the first coordinate, we get

$$(\iota_1)^{-1} \text{ad}(1, h_{-1}^{-1})\nu\theta\partial = \Omega(\pi_1 - t\pi_2)(j_1 + j_2) = \text{id} - t.$$

Thus, after using Remark 5.1.5 and hermitian stability, with the identification

$$kk^h(\Omega\Lambda, \Omega_{-1}M_2\Lambda) \cong kk^h(\Omega\Lambda, \Omega\Lambda),$$

the composition $\nu\theta\partial$ corresponds to $\text{id} - t$.

Because the $*$ -algebras involved in the argument above are flat, for any $B \in \text{Alg}_\ell^*$ we map apply the functor $- \otimes B$ of Example 3.2.29 to obtain the same identity in $kk^h(\Omega\Lambda B, \Omega\Lambda B)$.

Finally, apply the functor $kk^h(A, -)$ and the rest of the proof proceeds exactly as in [Kar80, Théorème 4.3]. \square

Corollary 5.3.9. *Let \mathfrak{C} and $H : \text{Alg}_\ell^* \rightarrow \mathfrak{C}$ be as in Proposition 3.2.22. The same argument as in Theorem 5.3.7 proves an analogous exact sequence for the groups obtained substituting $H(-)$ for $kk^h(A, -)$ in Definition 5.3.6.*

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