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Tesis de Licenciatura

Teorema de Representación de MV-Álgebras

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Introducción

Puisqu'on ne peut être universel en sachant tout ce qui se peut savoir sur tout, il faut savoir peu de tout. Car il est bien plus beau de savoir quelque chose de tout que de savoir tout d'une chose; cette universalité est la plus belle. Si on pouvait avoir les deux, encore mieux, mais s'il faut choisir, il faut choisir celle là...

Blaise Pascal: *Pensées, 37 [éd. Brunschvicg]*

... En aquel Imperio, el Arte de la Cartografía logró tal Perfección que el mapa de una sola Provincia ocupaba toda una Ciudad, y el mapa del Imperio, toda una Provincia. Con el tiempo, esos Mapas Desmesurados no satisficieron y los Colegios de Cartógrafos levantaron un Mapa del Imperio, que tenía el tamaño del Imperio y coincidía puntualmente con él. Menos Adictas al Estudio de la Cartografía, las Generaciones Siguientes entendieron que ese dilatado Mapa era inútil y no sin Impiedad lo entregaron a las Inclemencias del Sol y de los Inviernos. En los desiertos del Oeste perduran despedazadas Ruinas del Mapa, habitadas por Animales y por Mendigos; en todo el País no hay otra reliquia de las Disciplinas Geográficas.

Suárez Miranda: *Viajes de varones prudentes,* Libro Cuarto, Cap. XLV, Lérida, 1658.

Jorge Luis Borges: *Museo.*

Gejza Jenča demuestra en [17] que dado un MV-par (B, G), y una relación de equivalencia $\sim_G$, la effect álgebra $B/_{\sim_G}$ es una MV-effect álgebra. Nuestro principal objetivo es presentar esta demostración, con todos los resultados necesarios para su deducción.

En la sección 1.1 del capítulo 1, se estudian las propiedades básicas de las effect álgebras.
En la sección 1.2 se definen las propiedades de descomposición e interpolación de Riesz sobre effect álgebras; y damos una demostración de la proposición que dice que: "Toda effect álgebra que satisface la propiedad de descomposición de Riesz, cumple con la propiedad de interpolación de Riesz". También se demuestra que dada una effect álgebra que satisface la propiedad de descomposición de Riesz, al pasar al conjunto cociente vía una relación de congruencia de effect álgebras, dicho conjunto cociente también verifica la propiedad de descomposición de Riesz.
Luego, en la última sección del primer capítulo, se definen las propiedades elementales de las álgebras ϕ-simétricas, con el propósito de mostrar que en una effect álgebra con una estructura de reticulado, son equivalentes las propiedades de ϕ-simetría y de descomposición de Riesz.

En la sección 2.1 del capítulo 2, presentamos las propiedades elementales de las MV-álgebras, con el fin de dar una caracterización de las MV-álgebras como álgebras de Boole. En la sección 2.2 se definen las MV-effect álgebras, y probamos que toda MV-effect álgebra es una MV-álgebra (teorema 2.2.5), basándonos en los resultados publicados en los artículos de Chovanec y Kôpka [6] y [7]. Hacemos notar al lector, que en el libro [9] hay otra demostración del teorema 2.2.5, donde al demostrarse la propiedad asociativa correspondiente a las MV-álgebras, se supone erróneamente, que la suma entre los elementos tomados en consideración está siempre definida. Asimismo, en [10] y [11], se demuestra el teorema 2.2.5 en el contexto más general de las Pseudoeffect algebras.

En el capítulo tercero, se define la noción de MV-par (B, G), donde B es un álgebra de Boole, y G un subgrupo del grupo de automorfismos de B, que satisfacen ciertas condiciones. Dada $\sim_G$ una relación de equivalencia sobre B asociada a G, se demuestra que dado un MV-par (B, G), la effect álgebra resultante $B/{\sim_G}$, es una MV-effect álgebra, y en virtud del teorema 2.2.5, al que hicimos alusión en el párrafo anterior, es una MV-álgebra. Damos además, una caracterización de $B/{\sim_G}$ como álgebra de Boole, apoyándonos

en la caracterización de las MV-álgebras como álgebras de Boole mencionada más arriba al hacer mención de los contenidos del capítulo segundo.
Luego, y a modo de ejemplo, tomando un álgebra de Boole B finita con n átomos, y el grupo de automorfismos de B, demostramos que el conjunto $B/{\sim_G}$ es isomorfo a la MV-álgebra L_{n+1}. Además, considerando el álgebra de Boole de las partes finitas y cofinitas de los números naturales, y su grupo de automorfismos, probamos que el conjunto $B/{\sim_G}$ es isomorfo a $\Sigma(\mathbb{Z})$, conocida por ser el primer ejemplo de MV-álgebra no semisimple.

Por último indicamos que la demostración dada del teorema 2.2.5, en cuanto se refiere a la propiedad asociativa; el ejemplo 3.5, la caracterización dada en el Corolario 3.11, el ejemplo 3.13, así como las demostraciones de los ejemplos 3.2 y 3.12, las incluimos en el presente trabajo, y declaramos no haberlas visto en las publicaciones que hemos tenido a nuestro alcance.

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Abstract

Some properties of effect algebras, effect algebras with the Riesz descomposition property, ϕ-symmetric effect algebras, MV-algebras, Boolean algebras and MV-effect algebras are studied.
An MV-pair is a pair (B, G) where B is a Boolean algebra an G is a subgroup of the automorphism group of B satisfying certain conditions. Let $\sim_G$ be the equivalence relation on B naturally associated with G. For every MV-pair (B, G), the effect algebra $B/\sim_G$ is an MV-effect algebra, a proof of this fact is given. Moreover, we present a characterization of $B/\sim_G$ as a Boolean algebra.

Chapter 1

Effect Algebras

1.1 Basic notions

Definition 1.1.1 An *effect algebra* is a system $(E; \oplus, 0, 1)$ consisting of a set E with two special elements $0, 1 \in E$, called *zero* and the *unit*, an with a partially defined binary operation $\oplus$ satisfying the following conditions for all $p, q, r \in E$.

(E1) (Commutative Law) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(E2) (Associative Law) If $p \oplus q$ and $(p \oplus q) \oplus r$ are defined, then $q \oplus r$ and $p \oplus (q \oplus r)$ are defined and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

(E3) (Orthosuplementation Law) For every $p \in E$ there exist a unique $p' \in E$ such that $p \oplus p'$ is defined and $p \oplus p' = 1$.

(E4) (Zero-one Law) If $p \oplus 1$ is defined, then $p = 0$.

In an effect algebra, when we write an equation such as $p \oplus q = r$, we are asserting both that $p \oplus q$ is defined an that $p \oplus q = r$.

Example 1.1.2 Let $(B, 0, 1, \wedge, \vee, ^c)$ be a Boolean algebra, regarded as a bounded distributive lattice. Then $(B, 0, 1, \oplus)$ with $a \oplus b := a \vee b$ iff $a \wedge b = 0$, for all $a, b \in B$, is an effect algebra.

Example 1.1.3 [13] Let R be a (not necessarily commutative) ring with unity 1 and let E be the set of idempotents[1] in R. If $e, f \in E$, let $e \oplus f := e + f$ iff $ef = fe = 0$. Then $(E, 0, 1, \oplus)$ is an effect algebra.

Example 1.1.4 An *ordered Abelian group* is an Abelian group $(G; +, 0)$ equipped with a partial order $\leq$ which is translation-invariant, that is, given any $x, y, z \in G$, if $x \leq y$ then $x + z \leq y + z$. The *positive cone* of a partially ordered Abelian group G is the set G^+ of all positive elements in G. If G is a partially ordered Abelian group and $u \in G$, we define the interval

$$G^+[0, u] := \{g \in G : 0 \leq g \leq u\}.$$

If G is a partially ordered Abelian group an $u \in G^+$, then the interval $G^+[0, u]$ can be organized into an effect algebra $(G^+[0, u]; \oplus, 0, u)$ such that $p \oplus q$ is defined if and only if $p + q \leq u$, in wich case $p \oplus q = p + q$.

An effect algebra of the form $G^+[0, u]$, or isomorphic to such an effect algebra, is called an *interval effect algebra* with unit u of the group G.[2]

Lemma 1.1.5 Let E be an effect algebra and $p, q \in E$. Then:

(i) $p'' = p$.

(ii) $1' = 0$ and $0' = 1$.

(iii) For each $p \in E$, $p \oplus 0$ is defined and $p \oplus 0 = p$.

(iv) $0 \leq p \leq 1$ for all $p \in E$.

(v) If $p \oplus q$ is defined, then $q \oplus (p \oplus q)'$ is defined, and $p = (q \oplus (p \oplus q)')'$.

[1] An element f of a ring R is said to be idempotent if $f^2 = f$.

[2] An effect algebra with the Riesz descomposition property is an interval effect algebra, cf. [9].

(vi) (Cancellation Law) If $p \oplus r$ and $q \oplus r$ are defined and $p \oplus r = q \oplus r$, then $p = q$.

(vii) $p \oplus q = 0$ then $p = q = 0$.

Proof. (i): note that by (E1) and (E3), $p' \oplus p = p \oplus p' = 1$; hence, $p = p''$.

(ii): Since by (E3) $1 \oplus 1'$ is defined, (E4) implies that 1'=0, and by (i) we have that $0' = 1'' = 1$.

(iii): By (ii) $1 \oplus 0 = 1$; hence by (E3), (E1) and (E2):

$$1 = 1 \oplus 0 = (p' \oplus p) \oplus 0 = p' \oplus (p \oplus 0);$$

then by (E3) and (i) we conclude that $p \oplus 0 = p'' = p$.

(iv) Clearly by (iii) and (E_3).

(v): If $p \oplus q$ is defined, then by (E3) and (E2) we have that:

$$1 = (p \oplus q) \oplus (p \oplus q)' = p \oplus (q \oplus (p \oplus q)'),$$

and then (iv) follows from (E3) and (i).

(vi): Suppose that $p \oplus r = q \oplus r$, by (iv) and (E1) we have that:

$$p = (r \oplus (p \oplus r)')' = (r \oplus (q \oplus r)')' = q.$$

(vii): Finally suppose that $p \oplus q = 0$, then by (v) and (ii), $q \oplus (p \oplus q)' = q \oplus 1$, and by (E4), $q = 0$; hence by (iii) $0 = p \oplus 0 = p$. □

The binary relation $\leq$ defined on E by the prescription $p \leq q$ iff there is r such $p \oplus r = q$ is a partial order on E, called *the natural order of E.* Indeed, reflexivity follows from (iii) of Lemma 1.1.5, transitivity from (E2), and antisymmetry from (iii), (vi) and (vii) of Lemma 1.1.5.

Definition 1.1.6 The effect algebra E is *lattice ordered* iff, as a bounded partially ordered $\leq$ set $(E, \leq, 0, 1)$, it forms a lattice $(E, \leq, 0, 1, \wedge, \vee)$, i.e., $p \wedge q$ and $p \vee q$ exist for all $p, q \in E$.

Lemma 1.1.7 Let E be an effect algebra an let $p, q \in E$. Then:

(i) $p \leq q$ if and only if $q' \leq p'$.

(ii) $p \oplus q$ is defined if and only if $p \leq q'$.

Proof. (i) Suppose $p \leq q$, and take r such that $p \oplus r = q$. By (v) and (i) in Lemma 1.1.5, $p' = r \oplus (p \oplus r)' = r \oplus q'$, and this show that $q' \leq p'$. On the other hand, if $q' \leq p'$, by what we have just proved and (i) of Lemma 1.1.5, we have $p = p'' \leq q'' = q$.

(ii) Suppose first that $p \oplus q$, then by (v) in Lemma 1.1.5, $q' = p \oplus (p \oplus q)'$, hence $p \leq q'$. Suppose now that $p \leq q'$, i.e., that there is r such that $p \oplus r = q'$; then $1 = q \oplus q' = q \oplus (p \oplus r)$, hence by (E2) and (E1), $p \oplus q$ is defined. □

Definition 1.1.8 Let E be an effect algebra and $p, q \in E$. We say that p is *orthogonal* to q and write $p \perp q$ iff $p \leq q'$.

If $p, q \in E$ with $p \leq q$, there exist $r \in E$ with $p \perp r$ and $p \oplus r = q$. By the cancellation law, r is uniquely determined, and we can formulate the following definition:

Definition 1.1.9 If $p, q \in E$ with $p \leq q$, we define the *difference* $q \ominus p$ to be the unique element in E that satisfies $p \oplus (q \ominus p) = q$.

Proof of the next Lemma is omitted since it follow directly from Definition 1.1.9 and previously developed facts about effect algebras.

Lemma 1.1.10 Let E be an effect algebra and $p, q \in E$ with $p \leq q$. Then:

(i) $p = q$ if and only if $q \ominus p = 0$.

(ii) $p = 0$ if and only if $q \ominus p = q$.

(iii) $q \ominus p \leq q$ and $p = q \ominus (q \ominus p)$.

(iv) Let $r \in E$ such that $r \leq q \ominus p$, then:

$$p \leq q \ominus r$$

and,

$$(q \ominus p) \ominus r = (q \ominus r) \ominus p.$$

(v) $q \ominus p = (p \oplus q')'$.

(vi) $p \oplus q' = (q \ominus p)'$.

Proposition 1.1.11 (The De Morgan laws) Let E be an effect algebra, $p, q \in E$. Then

(i) If $p \wedge q$ exists in E, then $p' \vee q'$ exists in E and $(p \wedge q)' = p' \vee q'$.

(ii) If $p \vee q$ exists in E, then $p' \wedge q'$ exists in E and $(p \vee q)' = p' \wedge q'$.

Proof. Follows from Lemma 1.1.5 and Lemma 1.1.7. □

Definition 1.1.12 Let E and P be effect algebras. A mapping $\phi : E \to P$ is said to be

(i) a *morphism* iff satisfies the properties: $\phi(1_E) = 1_P$ and given $p, q \in E$ with $p \perp q$ then $\phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$;

(ii) a *homomorphism* iff ϕ is a morphism and $p, q \in E$ with $p \wedge q = 0$ implies $\phi(p) \wedge \phi(q) = 0$;

(iii) a *monomorphism* iff ϕ is a morphism and $p, q \in E$ with $\phi(p) \leq \phi(q) \Rightarrow p \leq q$; and,

(iv) an *isomorphism* iff ϕ is a surjective monomorphism.

Definition 1.1.13 Let E be an effect algebra. A relation $\sim$ on E is an *effect algebra congruence or a congruence on an effect algebra* iff the following conditions are satisfied.

(C1) $\sim$ is an equivalence relation.

(C2) If $p_1 \sim p_2$, $q_1 \sim q_2$ and $p_1 \oplus q_1$, $p_2 \oplus q_2$ exist, then $p_1 \oplus q_1 \sim p_2 \oplus q_2$.[3]

(C3) If $p \sim q \oplus r$, then there are q_1, r_1 such that $q_1 \sim q$, $r_1 \sim r$, $q_1 \oplus r_1$ exist and $p = q_1 \oplus r_1$.

(C4) If $p \sim q$, the $p' \sim q'$.

If $\sim$ is a congruence on an effect algebra E, we denote the equivalence class containing $p \in E$ by $[p]$ and denote the set of congruence classes by $E/_\sim$. We define $[p] \oplus [q]$ iff there exist $p_1, q_1 \in E$ such that $p_1 \sim p, q_1 \sim q$ and $p_1 \perp q_1$ and put $[p] \oplus [q] = [p_1 \oplus q_1]$. According to (C2), $[p] \oplus [q]$ is well defined. We say that [p] is less than or equal to [q] and write $[p] \leq [q]$ iff there exists an element $r \in E$ such that $[p] \perp [r]$ and $[p] \oplus [r] = [q]$.

Lemma 1.1.14 Let $\sim$ be a congruence on an effect algebra E. For all $p, q \in E$, the following are equivalent.

(i) [p]$\leq [q]$.

(ii) There is $p_1 \sim p$ such that $p_1 \leq q$.

(iii) There is $q_1 \sim q$ such that $p \leq q_1$.

Proof. $(ii) \Rightarrow (i)$ and $iii) \Rightarrow (i)$ are trivial.

$(i) \Rightarrow (ii)$: As $[p] \leq [q]$, there is $r \in E$ such that $[p] \oplus [r] = [q]$. This implies that there are $p_0, r_0 \in E$ such that $p_0 \sim p, r_0 \sim r, p_0 \perp r_0$, and $p_0 \oplus r_0 \sim q$. By the (C3) property, there are p_1, r_1 such that $p_1 \sim p_0, r_1 \sim r_0, p_1 \perp r_1$, and $p_1 \oplus r_1 = q$.

$(i) \Rightarrow (iii)$: Suppose $[p] \leq [q]$, then there exists $r \in E$ such that $[p] \oplus [r] = [q]$, then there are $p_0, r_0 \in E$ such that $p_0 \sim p, r_0 \sim r, p_0 \perp r_0$, and $p_0 \oplus r_0 \sim q$. By the (C3) property, there are p_1, r_1 such that

[3]A relation $\sim$ on E is a *weak congruence* iff (C1) and (C2) are satisfied.

$p_1 \sim p_0, r_1 \sim r_0, p_1 \perp r_1$, and $p_1 \oplus r_1 = q$. Then $p_1 \sim p$ and $p_1 \leq q$, and by Lemma 1.1.7 (i) $q' \leq p_1'$, hence there exists $v \in E$ such that $q' \oplus v = p_1'$. By (C4) property $[p_1'] = [p'] = [q' \oplus v] = [q'] \oplus [v]$, hence $[p'] \leq [q']$. As $(i) \Rightarrow (ii)$, there is $w \sim p'$ such that $w \leq p'$ and by Lemma 1.1.7 (i) $p \leq w'$. By (C4) property, $w \sim q'$ iff $w' \sim q$ and we can put $q_1 = w'$. □

Theorem 1.1.15 *If E is an effect algebra and $\sim$ is an effect algebra congruence, then $E/_\sim$ is an effect algebra.*

Proof. Clearly, $\oplus$ is commutative.

To prove associativity, assume that $[p] \oplus [q]$ and $([p] \oplus [q]) \perp [r]$. Then there exist $p_1, q_1 \in E$ such that $p_1 \sim p, q_1 \sim q, p_1 \perp q_1$ and $[p] \oplus [q] = [p_1 \oplus q_1]$. By definition $[r] \perp [p_1 \oplus q_1]$ so there exist $r_1, v \in E$ such that $r_1 \sim r$, $v \sim p_1 \oplus q_1, r_1 \perp v$ and

$$([p] \oplus [q]) \oplus [r] = [p_1 \oplus q_1] \oplus [r] = [v \oplus r_1]$$

Now $v \sim p_1 \oplus q_1, v \perp r_1$ imply by Lemma 1.1.7 (ii) that $v \sim p_1 \oplus q_1$ and $v \leq r_1'$, and by Lemma 1.1.14 there exists $r_2 \sim r_1'$ such that $p_1 \oplus q_1 \leq r_2$. By Lemma 1.1.7 (ii) we have that $p_1 \oplus q_1 \perp r_2'$ and by (C4) $r_2' \sim r_1$. Since $p_1 \perp q_1, r_2' \perp p_1 \oplus q_1$ we have by (E2) that $q_1 \perp r_2', [q] \perp [r]$ and $[q] \oplus [r] = [q_1 \oplus r_2']$. Moreover, $p_1 \perp (q_1 \oplus r_2')$ so $[p] \perp ([q] \oplus [r])$ and

$$([p] \oplus [q]) \oplus [r] = [v \oplus r_1] = [(p_1 \oplus q_1) \oplus r_2']$$

$$= [p_1 \oplus (q_1 \oplus r_2'] = [p_1] \oplus [q_1 \oplus r_2'] = [p] \oplus ([q] \oplus [r])$$

To show the orthosuplementation law, clearly $[p] \oplus [p'] = [p \oplus p'] = [1]$ for every $p \in E$. To prove the uniqueness of [p'] assume that $[p] \oplus [q] = [1]$. Then there exist $p_1, q_1 \in E$ such that $p_1 \sim p, q_1 \sim q$ and $p_1 \oplus q_1 \sim 1$. Also, there exists a $r \in E$ such that $p_1 \oplus q_1 \oplus r = 1$ so $q_1 \oplus r = p_1'$. Since $p_1 \oplus q_1 \sim p_1 \oplus q_1 \oplus r, r \perp p_1 \oplus q_1$ we have by Lema 1.1.7 (ii) that $p_1 \oplus q_1 \leq r'$, and by Lemma 1.1.14 there exists $v \sim r'$ such that $p_1 \oplus q_1 \oplus r \leq v$; then $p_1 \oplus q_1 \oplus r \perp v'$ and by (C4) $v' \sim r$. But then $v' = 0$ and hence $r \sim 0$. Now $q_1 \sim q, r \sim 0, q_1 \perp r$ imply by (C2) and (C4) that

$$q = q \oplus 0 \sim q_1 \oplus r = p_1' \sim p'$$

Hence, [q]=[p'].

Now, to prove the zero-one law, we assume that $[p] \perp [1]$. Then there exist $p_1, q \in E$ such that $p_1 \sim p, q \sim 1, p_1 \perp q$. Now $q \sim 1, q \perp p_1$ imply that there exists $p_2 \in E$ such that $p_2 \sim p_1, p_2 \perp 1$. Hence, $p_2 = 0$. Therefore, $p \sim p_1 \sim p_2$ so $[p] = [0]$. □

1.2 Effect algebras with the Riesz descomposition property

Definition 1.2.1 An *effect algebra E has the Riesz descomposition property* if, for $p_1, p_2, q_1, q_2 \in E, p_1 \oplus p_2 = q_1 \oplus q_2$ implies the existence of $\omega_{ij} \in E$ such that $p_i = \omega_{i1} \oplus \omega_{i2}$ and $q_j = \omega_{1j} \oplus \omega_{2j}$ for all $i, j \in \{1, 2\}$.

Lemma 1.2.2 Let E be an effect algebra. The following conditions are equivalent:

(i) For $p, q_1, q_2 \in E$ with $p \leq q_1 \oplus q_2$, there exist $p_1, p_2 \in E$ such that $p = p_1 \oplus p_2$ and $p_i \leq q_i$ $i = 1, 2$.

(ii) E has the Riesz descomposition property.

Proof. $(i) \Rightarrow (ii)$: Let $p_1, p_2, q_1, q_2 \in E$ and $p_1 \oplus p_2 = q_1 \oplus q_2$. Then we have $p_2 \leq q_1 \oplus q_2$. By (i), there exist $\omega_{11}, \omega_{12} \in E$ such that $p_1 = \omega_{11} \oplus \omega_{12}$ and each $\omega_{1j} \leq q_j$. Set $\omega_{2j} = q_j \ominus \omega_{1j}$ for each j, so $q_j = \omega_{1j} \oplus \omega_{2j}$. Since

$$p_1 \oplus p_2 = q_1 \oplus q_2 = \omega_{11} \oplus \omega_{21} \oplus \omega_{12} \oplus \omega_{22}$$

we also have $p_2 = \omega_{21} \oplus \omega_{22}$.

$(ii) \Rightarrow (i)$: Let $p, q_1, q_2 \in E$ and $p \leq q_1 \oplus q_2$. Set $v_1 = p$ and $v_2 = (q_1 \oplus q_2) \ominus p$, so that v_1, v_2 are elements such that $v_1 \oplus v_2 = q_1 \oplus q_2$. By (ii), there exist $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}$ such that each $v_i = \omega_{i1} \oplus \omega_{i2}$ and each $q_j = \omega_{1j} \oplus \omega_{2j}$. Then $p = v_1 = \omega_{11} \oplus \omega_{12}$. we also have $\omega_{1j} \leq \omega_{1j} \oplus \omega_{2j} = q_j$. □

Definition 1.2.3 *A partially ordered set X is said to have the Riesz interpolation property* if, for $p_1, p_2, q_1, q_2 \in E$ such that $p_i \leq q_j$ for all i, j, there exists $r \in E$ such that $p_i \leq r \leq q_j$ for all i, j.

Lemma 1.2.4 Let X be a partially ordered set.

(i) If X is a lattice, then X has interpolation.

(ii)If X is finite, bounded an has interpolation, then it is a lattice.

Proof. (i) Suppose X is a lattice an let $x, y, p, q \in X$. Then $x, y \leq p, q$ iff $x \vee y \leq p \wedge q$ and, if $x \vee y \leq p \wedge q$, then any element $z \in X$ with $x \vee y \leq z \leq p \wedge q$ satisfies $x, y \leq z \leq p, q$.

(ii)Let X be finite and bounded with interpolation property, let $p, q \in X$ an let $U := \{x \in X / x \leq p, q\}$. By induction on the number of elements in U, there exists $z \in X$ such that $z \leq p, q$ and $x \leq z$ for all $x \in U$. Thus, $z = p \wedge q$. A similar argument shows that any two elements $x, y \in X$ have a supremum $x \vee y$ in X. □

Proposition 1.2.5 Every effect algebra with the Riesz descomposition property has the Riesz interpolation property.

Proof. Suppose $p_1, p_2, q_1, q_2 \in E$ such that $p_i \leq q_j$ for all i, j. Let $q_1 \ominus p_1$, $q_1 \ominus p_2$, $q_2 \ominus p_1$ and $q_2 \ominus p_2$. We have that $q_2 \ominus p_1 \leq q_2 = p_2 \oplus (q_2 \ominus p_2)$, then Lemma 1.2.2 there exist $r_1, r_2 \in E$ such that $q_2 \ominus p_1 = r_1 \oplus r_2$ and $r_1 \leq p_2$, $r_2 \leq q_2 \ominus p_2$.

Let $p_2 \ominus r_1$ and $(q_2 \ominus p_2) \ominus r_2$. Thus,

$$q_2 = p_1 \oplus (q_2 \ominus p_1) = p_2 \oplus (q_2 \ominus p_2)$$

implies that:

$$p_1 \oplus r_1 \oplus r_2 = r_1 \oplus (p_2 \ominus r_1) \oplus r_2 \oplus ((q_2 \ominus p_2) \ominus r_2),$$

hence,

$$p_1 = (p_2 \ominus r_1) \oplus ((q_2 \ominus p_2) \ominus r_2),$$

by the cancellation law.

In the same way,

$$q_1 = p_1 \oplus (q_1 \ominus p_1) = p_2 \oplus (q_1 \ominus p_2),$$

implies that:

$$(p_2 \ominus r_1) \oplus ((q_2 \ominus p_2) \ominus r_2) \oplus (q_1 \ominus p_1) = (r_1 \oplus (p_2 \ominus r_1)) \oplus (q_1 \ominus p_2),$$

then, by cancellation law,

$$(q_2 \ominus p_2) \ominus r_2) \oplus (q_1 \ominus p_1) = r_1 \oplus (q_1 \ominus p_2).$$

By the Riesz descomposition property there exist $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} \in E$ such that:

$$q_1 \ominus p_1 = \omega_{11} \oplus \omega_{12},$$

$$(q_2 \ominus p_2) \ominus r_2 = \omega_{21} \oplus \omega_{22},$$

$$r_1 = \omega_{11} \oplus \omega_{21},$$

and

$$q_1 \ominus p_2 = \omega_{12} \oplus \omega_{22}.$$

Hence,

$$q_2 \ominus p_1 = r_1 \oplus r_2 = \omega_{11} \oplus \omega_{21} \oplus r_2,$$

and

$$q_2 \ominus p_2 = r_2 \oplus (q_2 \ominus p_2) \ominus r_2 = r_2 \oplus \omega_{21} \oplus \omega_{22}.$$

Also, $\omega_{11} \leq q_1 \ominus p_1$ and $q_1 \ominus p_1 \perp p_1$. By Lemma 1.1.7 $(q_1 \ominus p_1)' \leq \omega_{11}'$ and $p_1 \leq (q_1 \ominus p_1)'$, then $p_1 \leq \omega_{11}'$ by transitivity, and $p_1 \perp \omega_{11}$.

We have that $p_1 \leq p_1 \oplus \omega_{11} \leq p_1 \oplus (q_1 \ominus p_1) = q_1$. Also,

$$p_2 \oplus r_2 \oplus \omega_{21} \leq p_2 \oplus (q_2 \ominus p_2) = q_2 = p_1 \oplus (q_2 \ominus p_1) = p_1 \oplus \omega_{11} \oplus \omega_{21} \oplus r_2,$$

so $p_2 \leq p_1 \oplus \omega_{11}$ by the cancellation law. Then we conlude that:

$$p_1 \oplus \omega_{11} \leq p_1 \oplus (q_2 \ominus p_1) = q_2,$$

and

$$p_1, p_2 \leq p_1 \oplus \omega_{11} \leq q_1, q_2.$$

This completes the proof. □

The opposite implication of the last proposition is not true. The four-element effect algebra D, called *diamond*, consists of 0, two atoms[4] p, q with $p \neq q$ such that $p = p'$ and $q = q'$, and the unit $1 = p \oplus p' = q \oplus q'$, that trivially satisfies the Riesz interpolation property but it is not satisfies de Riesz descomposition property. If p and q are two atoms in the diamond D, then $p \oplus q$ is not defined because if $p \oplus q = q \oplus q$, so $p = q$ by the cancellation law, contradicting $p \neq q$.

Proposition 1.2.6 Let E be an effect algebra satisfying the Riesz descomposition property. If $\sim$ is an effect algebra congruence, then $E/\sim$ also satisfies the Riesz descomposition property.

Proof. Let Riesz descomposition property hold. Assume that:

$$[p] \oplus [q] = [r] \oplus [v].$$

Without loss of generality we may assume that $p \perp q$ and $r \perp v$, that is, $p \oplus q \sim r \oplus v$. By (C3), there are r_1, v_1 such that $r_1 \sim r$, $v_1 \sim v$, and $p \oplus q = r_1 \oplus v_1$. By the Riesz descomposition property, there are ω_{ij}, $i, j = 1, 2$ such that $p_i = \omega_{i1} \oplus \omega_{i2}$ $(i = 1, 2)$ and $q_j = \omega_{1j} \oplus \omega_{2j}$ $(j = 1, 2)$, and then $[p_i] = [\omega_{i1}] \oplus [\omega_{i2}]$ $(i = 1, 2)$, and $[q_j] = [\omega_{1j}] \oplus [\omega_{2j}]$ $(j = 1, 2)$. This completes the proof. □

1.3 Phi-Symmetric Effect Algebras

Proposition 1.3.1 (i) Let E be an effect algebra, $a, b, c \in E$, $a, b \leq c$. If $a \vee b$ exists in E, then $(c \ominus a) \wedge (c \ominus b)$ exists in E, and

$$c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$$

[4] A nonzero element $p \in E$ is called an atom if $E[0, p] = \{x \in E : 0 \leq x \leq p\} = \{0, p\}$.

In particular, if $a \perp b$ and we put $c = a \oplus b$, then

$$(a \oplus b) \ominus (a \vee b) = a \wedge b$$

(ii) Let E be a lattice ordered effect algebra, $a, b, c \in E$, $a \leq c, b \leq c$. Then

$$c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$$

In particular, if we put $c = a \vee b$,

$$(a \vee b) \ominus (a \wedge b) = ((a) \ominus a) \vee ((a \vee b) \ominus b)$$

(iii) In a lattice ordered effect algebra, for $c \leq a, b$

$$(a \wedge b) \ominus = (a \ominus c) \wedge (b \ominus c)$$

Proof. (i)From the inequalities

$$a \leq a \vee b \leq c,$$

$$b \leq a \vee b \leq c,$$

we have:

$$c \ominus (a \vee b) \leq c \ominus a$$

and

$$c \ominus (a \vee b) \leq c \ominus b.$$

For any other $\omega \in E$ with $\omega \leq c \ominus a, \omega \leq c \ominus b, a = c \ominus (c \ominus a) \leq c \ominus \omega$, $b = c \ominus (c \ominus b) \leq c \ominus \omega$, therefore,

$$a \vee b \leq c \ominus \omega \leq c,$$

and so

$$\omega = c \ominus (c \ominus \omega) \leq c \ominus (a \vee b),$$

wich implies that $c \ominus (a \vee b)$ is the greatest lower bound of the set $\{c \ominus a, c \ominus b\}$.

(ii) From the inequalities:

$$a \wedge b \leq a \leq c$$

and

$$a \wedge b \leq b \leq c$$

it follows that $c \ominus a \leq c \ominus (a \wedge b)$ and $c \ominus b \leq c \ominus (a \wedge b)$.
For $\omega \in E$ with $c \ominus a \leq \omega, c \ominus b \leq \omega$, then:

$$c \ominus a = (c \ominus a) \wedge c \leq \omega \wedge c \leq c \leq c,$$

which gives $c \ominus (\omega \wedge c) \leq a$, and similarly $c \ominus (\omega \wedge c) \leq b$, therefore:

$$c \ominus (\omega \wedge c) \leq a \wedge b.$$

Then we obtain $c \ominus (a \wedge b) \leq \omega \wedge c \leq \omega$ wich implies that $c \ominus (a \wedge b)$ is the least upper bound of the set $\{c \ominus a, c \ominus b\}$.

(iii) From $c \leq a \wedge b \leq a, b$ it follows that

$$(a \wedge b) \ominus c \leq (a \ominus c) \wedge (b \ominus c) \leq a \wedge b$$

If $\omega \in E$ is such that $\omega \leq a \ominus b, b \ominus c$, then $\omega \oplus c \leq a, b$, hence $\omega \leq (a \wedge b) \ominus c$. Hence $(a \wedge b) \ominus c$ is the greatest lower bound of $\{a \ominus c, b \ominus c\}$. □

Proposition 1.3.2 Let E be a lattice ordered effect algebra, $a, b \in E$. Then $((a \vee b) \ominus a) \wedge ((a \vee b) \ominus b) = 0$.

Proof. In Proposition 1.3.1 (i) put $c = a \vee b$. □

Proposition 1.3.3 Let E be a lattice ordered effect algebra, $c \leq a, c \leq b$. Then $(a \ominus c) \vee (b \ominus c) = (a \vee b) \ominus c$.

Proof. From $c \leq a \leq a \vee b, c \leq b \leq a \vee b$ we get

$$a \ominus c \leq (a \vee b) \ominus c, b \ominus c \leq (a \vee b) \ominus c.$$

Let $\omega \in E$ be such $a \ominus c, b \ominus c \leq \omega$. Then:

$$a \ominus c \leq \omega \wedge ((a \vee b) \ominus c) \leq (a \vee b) \ominus c$$

imply

$$((a \vee b) \ominus c) \ominus (\omega \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus c) \ominus (a \ominus c) = ((a \vee b) \ominus a);$$

and similarly,

$$((a \vee b) \ominus c) \ominus (\omega \wedge ((a \vee b) \ominus c)) \leq ((a \vee b) \ominus b).$$

Therefore,

$$((a \vee b) \ominus c) \ominus (\omega((a \vee b) \ominus c)) \leq ((a \vee b) \ominus a) \wedge ((a \vee b) \ominus b) = 0$$

by Proposition 1.3.2. Hence:

$$(a \vee b) \ominus c \leq \omega,$$

which gives the desired result. $\square$

Corollary 1.3.4 Let E be a lattice ordered effect algebra. Then:

$$(a \ominus (a \wedge b)) \wedge (b \ominus (a \wedge b)) = 0.$$

Proof. In Proposition 1.3.1(iii) put $c = a \wedge b$. $\square$

Proposition 1.3.5 Let E be a lattice ordered effect algebra. If $x \perp y$ and $x \perp z$, then

(i) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

(ii) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$.

Proof. (i) By Proposition 1.3.1 (iii),

$$((x \oplus y) \wedge (x \oplus z)) \ominus x = ((x \oplus y) \ominus x) \wedge ((x \oplus z) \ominus x) = y \wedge z,$$

whence:

$$(x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z).$$

(ii) By Proposition 1.3.3,

$$((x \oplus y) \vee (x \oplus z)) \ominus x = ((x \oplus y) \ominus x) \vee ((x \oplus z) \ominus x) = v \vee z,$$

whence:

$$(x \oplus y) \vee (x \oplus z) = x \oplus (y \vee z).$$

□

Definition 1.3.6[5] Let E be a lattice ordered effect algebra. The mapping

$$\phi : E \times E \to E$$

$$\phi(p, q) := p \ominus (p \wedge q') = [p' \oplus (p \wedge q')]'$$

is called the *Sasaki mapping* on E.

Theorem 1.3.7 (Parallelogram Theorem)[6] Let E be a lattice ordered effect algebra and $p, q \in E$. Then:

(i)

$$p = (p \wedge q) \oplus \phi(p, q')$$

and

$$q = (p \wedge q) \oplus \phi(q, p').$$

(ii)

$$p \vee q = p \oplus \phi(p, q)$$

$$= q \oplus \phi(q', p) = (p \wedge q) \oplus \phi(p, q') \oplus \phi(p', q)$$

$$= (p \wedge q) \oplus \phi(q, p') \oplus \phi(q', p).$$

[5]If E is an orthomodular lattice an $p \in E$, the *Sasaki projection* $\phi_p : E \to E$ is defined by $\phi_p(q) := p \wedge (p' \vee q)$ for all q. Thus, defining $a \oplus b := a \vee b$ exactly when $a \perp b$, $(\phi_p(q))' = p' \vee (p \wedge q') = p' \oplus (p \wedge q')$, so $\phi_p(q) = (p' \oplus (p \wedge q'))' = p \ominus (p \wedge q')$. This suggest this definition. It is well-known that in an orthomodular lattice, the Sasaki projection is a self-adjoint and idempotent residuated mapping. Recall that a mapping $\alpha : E \to E$ is a residuated if there is a mapping $\beta : E \to E$, called the residual of α , such that, for all $x, y \in E$, $\alpha(x) \leq y \Leftrightarrow x \leq \beta(y)$. A residuated mapping $\alpha : E \to E$ is called self-adjoint if its residual has the form $\beta(y) = [\alpha(y')]'$ for all $y \in E$. Evidently, is self-adjoint iff $\alpha(y) \perp y \Leftrightarrow x \perp \alpha(y)$ for all $x, y \in E$ [1].

[6]In the literature, there are various *parallelogram* rules, laws, or conditions involving the similarity, in one sense or the other, of the intervals $[p \wedge q, p]$ and $[q, p \vee q]$, or of the differences $(p \vee q) \ominus q$ and $p \ominus (p \wedge q)$ in a lattice. In our present context, these conditions can be studied in terms of the Sasaki mapping [1].

(iii)

$$\phi(p,q')\oplus\phi(p',q)=\phi(q,p')\oplus\phi(q',p)$$
$$=(p\vee q)\ominus(p\wedge q)=\phi(p',q)\vee\phi(q',p).$$

(iv)

$$\phi(p,q')\wedge\phi(q,p')=0$$

and

$$\phi(p',q)\wedge\phi(q',p)=0.$$

Proof. Part (i) follows from the facts that $\phi(p,q')=p\ominus(p\wedge q)$ and $\phi(q,p')=q\ominus(p\vee q)$.

For part (ii), there exists $k\in E$ with $p\vee q=p\oplus k$. Thus,

$$p\oplus k\oplus(p\vee q)'=1,$$

so

$$p\oplus(p'\wedge q')\oplus k=u,$$

and $k=\phi(p',q)$. By symmetry,

$$p\vee q=q\oplus\phi(q',p),$$

and the remaining parts of (ii) follow from (i).

All but the last equality in part (iii) follows from part (ii). To prove the las equality, note that, since $p\wedge q\leq p$, there exists $k\in E$ with

$$p\vee q=(p\wedge q)\oplus k.$$

Thus,

$$(p\wedge q)\oplus k\oplus(p\vee q)'=1,$$

so,

$$(p\wedge q)\oplus(p\vee q)'\oplus k=1,$$

and

$$k=\left[(p\wedge q)\oplus(p\vee q)'\right]'.$$

Then:

$$p \vee q = (p \wedge q) \oplus [(p \wedge q) \oplus (p \vee q)']' = (p \wedge q) \oplus [(p \wedge q) \oplus (p' \wedge q')]'$$

Also by proposition 1.3.5.

$$(p \wedge q) \oplus (p' \wedge q') = [p \oplus (p' \wedge q')] \wedge [q \oplus (p' \wedge q')] = [\phi(p', q)]' \wedge [\phi(q', p)]'$$

So:

$$p \vee q = (p \wedge q) \oplus [\phi(p', q) \vee \phi(q', p)]$$

that is,

$$(p \vee q) \ominus (p \wedge q) = \phi(p', q) \vee \phi(q', p)$$

(iv) The first part follows from Corollary 1.3.4, the second by symmetry. □

Definition 1.3.8 An effect algebra is ϕ-symmetric iff it is lattice ordered and $\phi(p, q) = \phi(q, p)$, i.e. $p \ominus (p \wedge q') = q \ominus (q \wedge p'),$ for all elements $p, q \in E$.

Theorem 1.3.9 For a lattice ordered effect algebra E, the following conditions are mutually equivalent:

(i) E is ϕ-symmetric.

(ii) $a, b, c \in E \Rightarrow a \ominus (a \wedge b) = (a \vee b) \ominus b$.

(iii) E has the Riesz descomposition property.

(iv) $x, y, z \in E$ with $y \perp z \Rightarrow x \wedge (y \oplus z) \leq (x \wedge y) \oplus (x \wedge z)$.

(v) $x, y, z \in E$ with $y \perp z \Rightarrow x \wedge (y \oplus z) \leq (x \wedge y) \oplus z$.

(vi) $a \leq (a \wedge b) \oplus (a \wedge b')$ for all $a, b \in E$.

(vii) $a, b \in E$ with $a \wedge b = 0 \Rightarrow a \perp b$.

Proof. That $(i) \Rightarrow (ii)$ follows immediately from the facts that $a \ominus (a \vee b) = \phi(a, b')$ and $\phi(b', a) = (a \vee b) \ominus b$ in Theorem 1.3.7.

To prove $(ii) \Rightarrow (iii)$, assume (ii) and suppose $a, b, c \in E$ with $b \perp c$ and $a \leq b \oplus c$. Then $a, b \leq b \oplus c$, so

$$a \vee b \leq b \oplus c$$

and

$$a \ominus (a \wedge b) = (a \vee b) \ominus b \leq (b \oplus c) \ominus b = c.$$

Let $b_1 := a \wedge b$ and $c_1 := a \ominus (a \wedge b)$. Then $b_1 \leq b, c_1 \leq c$, and $a = b_1 \oplus c_1$.

To prove $(iii) \Rightarrow (iv)$, assume (iii) and the hypotheses of (iv). Thus, since

$$x \wedge (y \oplus z) \leq y \oplus z,$$

there are elements $y_1 \leq y$ and $z_1 \leq z$ with

$$x \wedge (y \oplus z) = y_1 \oplus z_1.$$

Therefore, $y_1, z_1 \leq x \wedge (y \oplus z) \leq x$, so $y_1 \leq x \wedge y$ and $z_1 \leq x \wedge z$, and it follows that

$$x \wedge (y \oplus z) = y_1 \oplus z_1 \leq (x \wedge y) \oplus (x \wedge z)$$

That $(iv) \Rightarrow (v)$ is obvious.

To prove $(v) \Rightarrow (vi)$, assume(v) and let $a, b \in E$. Then, by (v),

$$a = a \wedge 1 = a \wedge (b \oplus b') \leq (a \wedge b) \oplus b',$$

so by (v) again,

$$a = a \wedge [b' \oplus (a \wedge b)] \leq (a \wedge b') \oplus (a \wedge b).$$

That $(vi) \Rightarrow (vii)$ is obvious.

Next we prove that $(vii) \Rightarrow (i)$. Assume (vii). Replacing q by q' in Theorem 1.3.7. we have

$$\phi(p,q) \oplus \phi(p',q') = \phi(q,p) \vee \phi(p',q')$$

by part (iii) and $\phi(p',q') \wedge \phi(q,p) = 0$ by part (iv). Therefore, by (vii), $\phi(p',q') \perp \phi(q,p)$, so

$$\phi(q,p) \vee \phi(p',q') = \phi(q,p) \oplus \phi(p',q')$$

by Proposition 1.3.1. part (i). Hence

$$\phi(p,q) \oplus \phi(p',q') = \phi(q,p) \oplus \phi(p',q'),$$

and

$$\phi(p,q) = \phi(q,p)$$

follows from the cancellation law. Therefore $(vii) \Rightarrow (i)$, and we have proved that Conditions (i) trought (vii) are mutually equivalent. □

Corollary 1.3.10 A finite effect algebra with the Riesz descomposition property is ϕ-symmetric.

Proof. Let F be a finite effect algebra with the Riesz descomposition property. By proposition 1.2.5, F has the interpolation, whence it is lattice ordered by Lemma 1.2.4. Consequently, F is ϕ-symmetric by Part (iii) of Theorem 1.3.9. □

1.4 Bibliographical remarks

As a general reference for this chapter, we mention the book [9].

An effect algebra is based on a partial binary operation $\oplus$. The operation $\oplus$ goes back to the original ideas or G. Boole [3], who supposed that $a + b$ will denote the logical disjunction and ab the logical conjunction of a,b, respectively. In fact, Boole only wrote $a+b$ when $ab = 0$, and this is all that is needed for probability theory: if $ab = 0$, then $P(a + b) = P(a) + P(b)$, where

P is a probability measure. Therefore, + can be introduced as a partially defined binary operation.

Effect algebras were introduced as abstraccions of the algebra of Hilbert-space effect operators, used in the study of the theory of measurement in quantum mechanics, by Foulis and Bennett in [14].

Lemma 1.1.14 appears in [17].

Theorem 1.1.15 is due to [15].

Lemma 1.2.2 is taken from [9], Lemma 2.1.4 from [1] and Proposition 2.1.6 from [8].

The third section of this chapter is based on the paper [1].

Chapter 2

MV-algebras

2.1 MV and Boolean algebras

2.1.1 Basic notions

Definition 2.1.1.1 An *MV-algebra* is an algebra $\mathcal{M} = (M, +, *, 0, 1)$, where M is a nonempty set, 0 and 1 are constants, + is a total binary operation, and $*$ is a unary operation satisfying the following axioms.

(MV1) $(a + b) + c = a + (b + c)$.

(MV2) $a + b = b + a$.

(MV3) $a + 0 = a$

(MV4) $(a^*)^* = a$.

(MV5) $a + 1 = 1$.

(MV6) $(a^* + b)^* + b = (a + b^*)^* + a$.

(MV7) $a + a^* = 1$.

(MV8) $0^* = 1$.

Example 2.1.1.2 A singleton $\{0\}$ is a trivial example of an MV-algebra.

Example 2.1.1.3 If $(B, 0, 1, \wedge, \vee, ^c)$ is a Boolean algebra, then $(B, \vee, ^c, 0, 1)$ is an MV-algebra, where $\vee$, ^{c}, 0 and 1 denote, respectively, the join, the complement, the smallest and the greatest elements in B.

Example 2.1.1.4 Consider the real unit interval:
$[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, and for all $x, y \in [0, 1]$, let $x+y := min\{1, x+y\}$ and $x^* := 1 - x$. It is easy to see that $[0, 1] := ([0, 1], +, ^*, 0, 1)$ is an MV-algebra.

Example 2.1.1.5 A subalgebra of an MV-algebra M is a subset S of M containing the zero element of M, closed under the operations of M, an equipped with the restriction to S of these operations. The rational numbers in $[0, 1]$, and, for each integer $n \geq 2$, the n-element set

$$L_n := \{0, 1/(n-1), ..., (n-2)/(n-1), 1\}$$

yield examples of subalgebras of $[0, 1]$.

Example 2.1.1.6 Given an MV-algebra M and a set X, the set M^X of all functions $f : X \to M$ becomes an MV-algebra if the operations $+$ and * and the element 0 are defined pointwise. The continous functions from $[0, 1]$ into $[0, 1]$ form a subalgebra or the MV-algebra $[0, 1]^{[0,1]}$.

Definition 2.1.1.7 On each MV-algebra M we define the operations $\circ$ and - as follows:

$$x \circ y := (x^* + y^*)^*$$

$$x - y := x \circ y^*$$

As a consequence of (MV4), we can write:

(MV9) $x + y = (x^* \circ y^*)^*$.

Axiom (MV6) can now be written as:

(MV6') $(x - y) + y = (y - x) + x$.

Note that in the MV-algebra $[0, 1]$ we have $x \circ y = max(0, x + y - 1)$ and $x - y = max(0, x - y)$.

Lemma 2.1.1.8 Let M be an MV-algebra an $x, y \in M$. Then the following conditions are equivalent:

(i) $x^* + y = 1$.

(ii) $x \circ y^* = 0$.

(iii) $y = x + (y - x)$.

(iv) There is an element $z \in M$ such that $x + z = y$.

Proof. $(i) \Rightarrow (ii)$: By (MV4) and (MV7).

$(ii) \Rightarrow (iii)$: Immediate from (MV3) and (MV6').

$(iii) \Rightarrow (iv)$: Take $z = y - x$.

$(iv) \Rightarrow (i)$: By (MV9) and (MV7), $x^* + x + z = 1$. □

Let M be an MV-algebra. For any two elements x and y of M let us agree to write

$$x \leq y$$

iff x and y satisfy the above equivalent conditions (i)-(iv). It follows that $\leq$ is a partial order, called the *natural order* of M. Indeed, reflexivity is equivalent to (MV7), antisymmetry follows from conditions (ii) and (iii), and transitivity follows from condition (iv).

An MV-algebra whose natural order is total is called an *MV-chain.*

Lemma 2.1.1.9 In every MV-algebra M the natural order $\leq$ has the following properties:

(i) $x \leq y$ if and only if $y^* \leq x^*$.

(ii) If $x \leq y$ then for each $z \in M$, $x + z \leq y + z$ and $x \circ z \leq y \circ z$.

(iii) $x \circ y \leq z$ if and only if $x \leq y^* + z$.

Proof. (i): This follows from Lemma 2.1.1.8 (i), since $x^* + y = y^{**} + x^*$.

(ii): The monotonicity of + is an easy consequence or Lemma 2.1.1.8 (iv); using (i), one immediately proves the monotonicity of $\circ$.

(iii): It is sufficient to note that $x \circ y \leq z$ is equivalent to
$1 = (x \circ y)^* + z = x^* + y^* + z$. □

Proposition 2.1.1.10 On each MV-algebra M the natural order determines a lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by

(i) $x \vee y = (x \circ y^*) + y = (x - y) + y$,

(ii) $x \wedge y = (x^* \wedge y^*)^* = x \circ (x^* + y)$.

Proof. To prove (i), by (MV6') and Lemma 2.1.1.8, $x \leq (x - y) + y$ and $y \leq (x - y) + y$. Suppose $x \leq z$ and $y \leq z$. By (i) and (iii) in Lemma 2.1.1.8, $x^* + z = 1$ and $z = (z - y) + y$. Then by (MV6') we can write

$$((x - y) + y)^* + z = ((x - y)^* - y) + y + (z - y)$$

$$= (y - (x - y)^*) + (x - y)^* + (z - y)$$

$$= (y - (x - y)^*) + x^* + y + (z - y)$$

$$= (y - (x - y)^*) - x^* + z = 1.$$

It follows that $(x - y) + y \leq z$, which completes the proof of (i). We now immediately obtain (ii) as a consequence or (i) together with Lemma 2.1.1.9 (i). □

Proposition 2.1.1.11 The following equations hold in every MV-algebra:

(i) $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$,

(ii) $x + (y \wedge z) = (x + y) \wedge (w + z)$.

Proof. By Lemma 2.1.1.9 (ii), $x \circ y \leq x \circ (y \vee z)$ and $x \circ z \leq x \circ (y \vee z)$. Suppose $x \circ y \leq t$ and $x \circ z \leq t$. Then by Lemma 2.1.1.9 (iii), $y \leq x^* + t$ and $z \leq x^* + t$, whence $y \vee z \leq x^* + t$. One more application of Lemma 2.1.1.9 (iii) yields $(y \vee z) \circ x \leq t$, which completes the proof of (i). It is now easy tosee that (ii) es a consequence of (i), using Lemma 2.1.1.9 (i), together with (MV4) and (MV9). □

Proposition 2.1.1.12 Every MV-algebra satisfies the equation

$$(x - y) \wedge (y - x) = 0$$

Proof. By making repeated use of (MV6) and its variants, together with the basic properties of the operations + and $\circ$ we obtain:

$$(x - y) \wedge (y - x)$$

$$= (x - y) \circ ((x - y)^* + (y - x))$$

$$= x \circ y^* \circ (y + x^* + (y - x)$$

$$= x \circ (x^* + (y - x)) \circ ((x^* + (y - x))^* + y^*)$$

$$= (y - x) \circ ((y - x)^* + x) \circ ((x^* + (y - x))^* + y^*)$$

$$= y \circ x^* \circ ((y - x)^* + x) \circ ((x \circ (y - x)^*) + y^*)$$

$$= x^* \circ (x + (y - x)^*) \circ y \circ (y^* + (x \circ (y^* + x)))$$

$$= x^* \circ (x + (y - x)^*) \circ (x \circ (y^* + x)) \circ ((x \circ (y^* + x))^* + y) = 0,$$

since by (MV7) and (MV9), $x^* \circ x = 0$. □

Let M be an MV-algebra. For each $x \in M$, we let $0x = 0$, and for each integer $n \geq 0$, $(n+1)x = nx + x$.

Lemma 2.1.1.13 Let x and y be elements of an MV-algebra M. If $x \wedge y = 0$ then for each integer $n \geq 0$, $nx \wedge ny = 0$.

Proof. If $x \wedge y = 0$ then by monotonicity (Lemma 2.1.1.9) and distributivity (Proposition 2.1.1.11),

$$x = x + (x \wedge y) = (x + x) \wedge (x + y) \geq 2x \wedge y,$$

whence:

$$0 = x \wedge y \geq 2x \wedge y,$$

in the same way $y \geq 2y \wedge x$. Then:

$$0 = x \wedge y \geq 2x \wedge 2y \wedge x \wedge y = 2x \wedge 2y.$$

It follows that $0 = 2x \wedge 2y$ and similarly:

$$0 = 4x \wedge 4y \wedge 4x = 8x \wedge 8y = \ldots.$$

The desired conclusion now follows from:

$$nx \wedge ny \leq 2^n x \wedge 2^n y = 0.$$

□

2.1.2 Homomorphism and ideals

Let M and N be MV-algebras. A function $f : M \to N$ is said to be a *homomorphism* iff $f(0) = 0$, for each $x, y \in M$ $f(x + y) = f(x) + f(y)$, and $f(x^*) = f(x)^*$. Following current usage, if f is one-one we shall equivalently say that f is an *injective* homomorphism, or an *embedding*. If the homomorphism $f : M \to N$ is onto N we say that f is *surjective*. By *isomorphism* we shall mean a surjective one-one homomorphism.

The *kernel* of a homomorphism $f : M \to N$ is the set

$$Ker(f) := f^{-1}(0) = \{x \in M : f(x) = 0\}$$

An *ideal* of an MV-algebra M is a subset I of M satisfyin the following conditions:

(I1) $0 \in I$,

(I2) If $x \in I$, $y \in M$ and $y \leq x$ then $y \in I$,

(I3) If $x \in I$ and $y \in I$ then $x + y \in I$.

The intersection of any family of ideals of M is still an ideal of M. For every subset $W \subseteq M$, the intersection of all ideals $I \supseteq W$ is said to be the ideal *generated* by W, and will be denoted $\langle W \rangle$.

The proof of the next lemma is immediate, and will be ommitted.

Lemma 2.1.2.1 Let W be a subset of an MV-algebra M. If $W = \emptyset$, then $\langle W \rangle = \{0\}$. If $W \neq \emptyset$, then

$$\langle W \rangle = \{x \in M : x \leq \omega_1 + ... + \omega_k, \omega_1, ..., \omega_k \in W\}$$

In particular, for each element z of an MV-algebra M, the ideal $\langle z \rangle = \langle \{z\} \rangle$ is called the *principal ideal generated by* z, and we have

$$\langle z \rangle = \{x \in M : nz \geq x, n \in \mathbb{N}_0\}$$

Note that $\langle 0 \rangle = \{0\}$ and $\langle 1 \rangle = M$. Further, for every ideal J of an MV-algebra M and each $z \in M$ we have

$$\langle J \cup \{z\} \rangle = \{x \in M : x \leq nz + a, n \in \mathbb{N}, a \in J\}$$

An ideal I of an MV-algebra M is *proper* iff $I \neq M$. We say that I es *prime* iff it es proper and satisfies the following condition:

(I4) For each x and y in M, either $(x - y) \in I$ or $(y - x) \in I$.

An ideal I of an MV-algebra M is called *maximal* iff it is proper and for each ideal $J \neq I$, if $I \subseteq J$ then $J = M$.

We denote by $\mathcal{I}(M)$, $\mathcal{P}(M)$ and $\mathcal{M}(M)$ the sets of ideals, prime ideals and maximal ideals of M respectively.

We omit the straighforward proof of the followings statements.

Lemma 2.1.2.2 Let M, N be MV-algebras, and $f : M \to N$ a homomorphism. Then the following properties hold:

(i) $f(x) \leq h(y)$ iff $x - y \in Ker(f)$.

(ii) f is injective iff $Ker(f) = \{0\}$.

(iii) $Ker(f) \neq M$ iff N is nontrivial[1] .

(iv) $Ker(f) \in \mathcal{P}(M)$ iff N is nontrivial and the image $f(M)$, as a subalgebra of N, is an MV-chain.

The following function d plays the role of a distance function in MV-algebras.

Definition 2.1.2.3 The distance function $d : M \times M \to M$ is defined by

$$d(x, y) := (x - y) + (y - x)$$

In the MV-algebra $[0, 1]$, $d(x, y) = |x - y|$. In every Boolean algebra the distance function coincides with the symmetric difference operation.

Proposition 2.1.2.4 In every MV-algebra M we have:

(i) $d(x, x) = 0$.

(ii) If $d(x, y) = 0$ then $x = y$.

(iii) $d(x, y) = d(y, x)$.

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

[1] An MV-algebra M is said *nontrivial* iff its universe M has more than one element.

(v) $d(x,y) = d(x^*, y^*)$.

(vi) $d(x+s, y+t) \leq d(x,y) + d(s,t)$.

Proof.(i), (iii) and (v): Immediately follow by definition.

(ii): Follows from the fact that $x+y=0$ implies $x=0=y$ and by Lemma 2.1.1.8 (iii).

(iv): Note first that

$$(x-z)^* + (x-y) + (y-z) = (x^* \vee y^*) + (z \vee y) \geq y^* + y = 1.$$

Hence,

$$(x-z) \leq (x-y) + (y-z).$$

In an entirely similar fashion:

$$(z-x) \leq (y-x) + (z-y),$$

whence (iii) follows from de monotonicity if + (Lemma 2.1.1.9).

(vi): In the same way that the proof of (iv), note that:

$$((x+s) - (y+t))^* + (x-y) + (s-t)$$

$$= (x+s)^* + (x \vee y) + (t \vee x) \geq (x+s)^* + x + s = 1.$$

□

As an immediate consequence we have:

Proposition 2.1.2.5 Let I be an ideal of an MV-algebra M. Then the binary relation $\equiv_I$ on M defined by $x \equiv_I y$ iff $d(x,y) \in I$ is a congruence relation. (Stated otherwise, $\equiv_I$ is an equivalence relation such that $x \equiv x$ and $y \equiv_I t$ imply $x^* \equiv_I s^*$ and $x+y \equiv_I s+t$.) Moreover, $I = \{x \in M : x \equiv_I 0\}$. Conversely, if $\equiv_I$ is a congruence on M, then $\{x \in M : x \equiv_I 0\}$ is an ideal, and $x \equiv_I y$ iff $d(x,y) = 0$.
Therefore, the correspondence $I \rightarrow \equiv_I$ is a bijection from the set of ideals of M onto the set of congruences on M.

Given $x \in M$, the equivalence class of x with respect to $\equiv_I$ will be denoted by x/I, and the quotient set $M/\equiv_I$ by M/I. Since $\equiv_I$ is a congruence, defining on the set M/I the operations

$$(x/I)^* := x^*/I$$

and

$$x/I + y/I := (x+y)/I,$$

the system $(M/I, +, ^*, 0/I)$ becomes an MV-algebra, called the *quotient algebra of M by ideal I.* Moreover, the correspondence $x \to x/I$ defines a homomorphism f_I from M onto the quotient algebra M/I, which is called the *natural homomorphism from M onto M/I.* Note that $Ker(f_I) = I$.

The proof of the following lemma is straighforward.

Lemma 2.1.2.6 If M, N, and S are MV-algebras, and $f : M \to N$ and $g : M \to S$ are surjective homomorphisms, then $Ker(f) \subseteq Ker(g)$ if and only if there is a surjective homomorphism $h : N \to S$ such that $h \circ f = g$, i.e., $h(f(x)) = g(x)$ for all $x \in M$. This homomorphism h is an isomorphism if and only if $Ker(f) = Ker(g)$.

As an inmediate consequence we have

Theorem 2.1.2.7 Let M and N ve MV-algebras. If $h : M \to N$ is a surjective homomorphism, then there is an isomorphism $f : M/Ker(h) \to N$ such that $f(x/Ker(h)) = h(x)$ for all $x \in M$.

Proposition 2.1.2.8 Let J be an ideal of an MV-algebra. For every $a \in M \backslash J$ there is a prime ideal P of M such that $J \subseteq P$ and $a \notin P$.

Proof. A routine application of Zorn's Lemma shows that there is an ideal I of M which is maximal with respect to the property that $J \subseteq I$ and $a \notin I$. We shall show that I is a prime ideal. Let x and y be elements of M, and suppose that both $x - y \notin I$ and $y - x \notin I$. By the maximality assumption the ideal generated by I and $x - y$ must contain the element a, then $a \leq s + p(s - y)$ for some $s \in I$ and some integer $p \geq 1$. Similarly, there is an element $t \in I$ and an integer $q \geq 1$ such that $a \leq t + q(y - x)$. Let $u = s + t$ and $n = max(p, q)$. Then $u \in I$, $a \leq u + n(x - y)$ and

$a \leq u + n(y - x)$. Hence by Proposition 2.1.1.12, Proposition 2.1.1.11 (ii) and Lemma 2.1.1.13, we have:

$$a \leq (u + n(x - y)) \wedge (u + n(y - x)) = u + (n(x - y) \wedge n(y - x)) = u,$$

whence $a \in I$, a contradiction. □

Corollary 2.1.2.9 Every proper ideal of an MV-algebra is an intersection of prime ideals.

2.1.3 Subdirect representation theorem

Let Γ a non empty set. The *direct product* of a family $\{M_i\}_{i\in\Gamma}$ of MV-algebras denoted by $\Pi_{i\in\Gamma}M_i$, is the MV-algebra obtained by endowing the set-theoretical cartesian product of the family with the MV-operations defined pointwise. In other words, $\Pi_{i\in\Gamma}M_i$ is the set of all functions $f : \Gamma \to \bigcup_{i\in\Gamma} M_i$ such that $f(i) \in M_i$, for all $i \in \Gamma$, with the operations * and $+$ defined by

$$f^*(i) := f(i)^*$$

and

$$(f + g)(i) := f(i) + g(i)$$

The zero element of $\Pi_{i\in\Gamma}M_i$ is the fuction $i \in \Gamma \to 0_i \in M_i$. For each $j \in \Gamma$, the map $\pi_j : \Pi_{i\in\Gamma}M_i \to M_j$ is defined by

$$\pi_j(f) := f(j).$$

Each π_j is a homomorphism onto M_j called the j^{th} *projection function.* In particular, for each MV-algebra M and a nonempty set X, the MV-algebra M^X is the direct product of the family $\{M_x\}_{x\in X}$, where $M_x = M$ for all $x \in X$.

Definition 2.1.3.1 An MV-algebra M is a *subdirect product* of a family $\{M_i\}_{i\in\Gamma}$ of MV-algebras if and only if there exists a one-one homomorphism $h : M \to \Pi_{i\in\Gamma}M_i$ such that, for each $j \in \Gamma$, the composite map $\pi_j \circ h$ is an homomorphism onto M_j.

The following result is a particular case of a theorem of Universal Algebra, due to Birkhoff [2].

Theorem 2.1.3.2 *An MV-algebra M is a subdirect product of a family $\{M_i\}_{i\in\Gamma}$ of MV-algebras if and only if there is a family $\{J_i\}_{i\in\Gamma}$ of ideals of M such that*

(i) $M_i \cong M/J_i$ for each $i \in \Gamma$,

and

(ii)$\bigcap_{i\in\Gamma} = \{0\}$.

Proof. Supposing first that M is a subdirect product of a family $\{M_i\}_{i\in\Gamma}$ of MV-algebras, let $h : M \to \Pi_{i\in\Gamma}M_i$ be a one-one homomorphism as given by Definition 2.1.3.1; for each $j \in \Gamma$, let $J_j = Ker(\pi_j \circ h)$. By Theorem 2.1.2.7, $M_j \cong M/J_j$. If $x \in \bigcap_{i\in\Gamma} J_i$, then $\pi_j(h(x)) = 0$ for all $j \in \Gamma$. This implies $h(x) = 0$, and since h is injective, $x = 0$. Therefore $\bigcap_{i\in\Gamma} J_i = \{0\}$, and conditions (i) and (ii) hold true.

Conversely, Suppose $\{J_i\}_{i\in\Gamma}$ to be a family of ideals of M satisfying conditions (i) and (ii). Let ϵ_i be an isomorphism of M/J_i onto M_i, as given by condition (i). Let ghe function $h : M \to \Pi_{i\in\Gamma}M_i$ be defined as follows: for each $x \in M$, $(h(x))(i) = \epsilon_i(x/J_i)$. It follows from (ii) that $Ker(h) = \{0\}$, whence, by Lemma 2.1.2.2 (ii), h is injective. Since for each $i \in \Gamma$ the map $a \in M \to a/J_i \in A/J_i$ is surjective, then $\pi_i \circ h$ maps M onto M_i. Thus, M is a subdirect product of the family $\{M_i\}_{i\in\Gamma}$, as required. □

Theorem 2.1.3.3 (Chang's Sufdirect Representation Theorem) *Every nontrivial MV-algebra s a subdirect product of MV-chains.*

Proof. By Theorem 2.1.3.2 and Lemma 2.1.2.2 (iv), an MV-algebra M is a subdirect product of a family of MV-chains if an only if there is a family $\{P_i\}_{i\in\Gamma}$ of prime ideals of M such that $\bigcap_{i\in\Gamma} P_i = \{0\}$. Now apply Corollary 2.1.2.9 to the ideal $\{0\}$. □

2.1.4 MV-equations

As we shall see, an important consequence of Chang's Subdirect Representation Theorem is that in order to prove that an equation holds in all MV-algebras it is sufficient to check that the equation holds in al MV-chains. To give a precise formulation to this result we shall now develop the necessary syntactic machinery.

Definition 2.1.4.1 By a *string* (or, *word*) over a nonempty set S we understand a finite list of elements of S.

For each natural number $t \geq 1$, let $S_t := \{0,^*, +, x_1, ..., x_t, (,)\}$. An *MV-term* in the variables $x_1, ..., x_t$ is a string over S_t arising from a finite numbber of applications of the following rules:

(T1) The elements 0 and x_i, for $i = 1, ..., t$, considered as one-element strings, are MV-terms.

(T2) If the string τ is an MV-term, then so is τ^*.

(T3) If the strings τ and σ are MV-terms, then so is $(\tau + \sigma)$.

In other wors, a string τ over S_t is an MV-term if and only if there is a finite list of strings over S_t, say $\tau_1, \tau_2, ..., \tau_n$, such that $\tau_n = \tau$ and for each $i \in \{1, ..., n\}$, τ_i satisfies at least one of the following conditions:

(i) $\tau_i = 0$ or $\tau_i = x_j$, for some $1 \leq j \leq t$,

(ii) there is $j < i$ such that $\tau_i = \tau_j^*$,

(iii) there are $j < i$ and $k < i$ such that $\tau_i = (\tau_j + \tau_k)$.

Those strings τ_i that belong to every formation sequece for τ are said to be the *subterms* of τ.

The following result is known as the *unique readability* theorem; its proof es precisely the same as for the classical propositional calculus, and is left as exercise.

Theorem 2.1.4.2 *Every term τ_i in the variables $x_1, ..., x_n$ satisfies precisely one of the above conditions (i)-(iii). Moreover, both term τ_j of case (ii) and the pair (τ_j, τ_k) are uniquely determined.*

We shall henceforth write $\tau(x_1, ..., x_2)$ to signify that τ is an MV-term in te variables $x_1, ..., x_2$.

Definition 2.1.4.3 Let M be an MV-algebra, τ an MV-term in the variables $x_1, ..., x_2$ and assume $a_1, ..., a_t$ are elements of M. Substituting an element $a_i \in M$ for all occurrences of the variable x_i in τ, for $i = 1, ..., t$, using the unique readatability theorem, and interpreting the symbols $0, +$ and * as the corresponding operations in M, we obtain an element of M, denoted $\tau^M(a_1, ..., a_t)$.

In more detail, proceeding by induction on the number of operation symbols occurring in τ, we define $\tau^M(a_1, ..., a_t)$ as follows:

(i) $x_i^M = a_i$, for each $i = 1, ..., t$;

(ii) $(\sigma^*)^M = (\sigma^M)^*$;

(iii) $(\sigma + \rho)^M = (\sigma^M + \rho^M)$.

Definition 2.1.4.4 An *MV-equation* (for short, an *equation*) in the variables $x_1, ..., x_t$ is a pair (τ, σ) of MV-terms in the variables $x_1, ..., x_t$.

Following tradition, we shall write $\tau = \sigma$ instead of (τ, σ). An MV-algebra M satisfies the MV-equation $\tau = \sigma$, in symbols,

$$M \models \tau = \sigma,$$

iff

$$\tau^M(a_1, ..., a_t) = \sigma^M(a_1, ..., a_t)$$

for all $a_1, ..., a_t \in M$.

Axioms (MV1)-(MV6) are examples of MV-equations, by definition, these equations are satisfied by all MV-algebras.

Lemma 2.1.4.5 Let M, N, M_i (for all $i \in \Gamma$) be MV-algebras:

(i) If $M \models \tau = \sigma$ the $S \models \tau = \sigma$ for each subalgebra S of M.

(ii) If $h : M \to N$ is a homomorphism, then for each MV-term τ in the variables $x_1, ..., x_s$ and each s-tuple $(a_1, ..., a_s)$ of elements of M we have $\tau^N(h(a_1), ..., h(a_s)) = h(\tau^M(a_1, ..., a_s))$. In particular, when h maps M onto N, from $M \models \tau = \sigma$ it follows that $N \models \tau = \sigma$.

(iii) If $M_i \models \tau = \sigma$ for each $i \in \Gamma$, then $\Pi_{i \in \Gamma} M_i \models \tau = \sigma$.

Proof. Conditions (i) and (ii) are immediate.

(iii): Let $f_1, ..., f_s \in M = \Pi_{i \in \Gamma} M_i$. By hypothesis, for each $i \in \Gamma$ we can write

$$\tau^M(f_1, ..., f_s)(i) = \tau^{M_i}(f_1(i), ..., f_s(i)) =$$

$$\sigma^{M_i}(f_1(i), ..., f_s(i)) = \sigma^M(f_1, ..., f_s)(i),$$

whence $\tau^M(f_1, ..., f_s) = \sigma^M(f_1, ..., f_s)$. □

Theorem 2.1.4.6 *Let M be the subdirect product of a family $\{M_i\}_{i \in \Gamma}$ of MV-algebras; let $\tau = \sigma$ be an MV-equation. Then $M \models \tau = \sigma$ if and only if $M_i \models \tau = \sigma$ for each $i \in \Gamma$.*

Proof. Let $h : M \to \Pi_{i \in \Gamma} M_i$ be a one-one homomorphism as given by Definition 2.1.3.1. Suppose that $M \models \tau = \sigma$. Since for each $i \in \Gamma$, $\pi \circ h$ maps M onto M_i, it follows that $M_i \models \tau = \sigma$.

Conversely, suppose that $M_i \models \tau = \sigma$ for all $i \in \Gamma$. By the above Lemma, $\Pi_{i \in \Gamma} M_i \models \tau = \sigma$, and since $h(M)$ is a subalgebra of $\Pi_{i \in \Gamma} M_i$, $h(M) \models \tau = \sigma$. Since h^{-1} maps $h(M)$ onto M, we conclude that $M \models \tau = \sigma$. □

Corollary 2.2.4.7 An MV-equation is satisfied by all MV-algebras if and only if it is satisfied by all MV-chains.

Proof. Suppose that $\tau = \sigma$ is satisfied by all MV-chains, an let M be an MV-algebra. If $M = \{0\}$ the, trivially, $\tau^M(0, ..., 0) = 0 = \sigma^M(0, ..., 0)$, whence $M \models \tau = \sigma$. If M is nontrivial the desired conclusion follows from Theorems 2.1.3.3 and 2.1.4.6. ☐

2.1.5 Boolean algebras

We assume that the reader is familiar with the fundamental notions from Boolean algebras.[2]

We have already noted that Boolean algebras are particular cases of MV-algebras. In this section we shall caracterize Boolean algebras among MV-algebras.

The natural order makes every MV-algebra M into a lattice with a minimum element 0 and maximum 1. We shall denote this lattice by

$$L(M)$$

Recall that the lattice operations of join and meet are definable via the MV-operations by the following formulas:

$$x \vee y = (x \circ y^*) + y = (x - y) + y$$

$$x \wedge y = (x^* \vee y^*) = x \circ (x^* + y)$$

A lattice is called *distributive* iff the following distributive laws hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

When M is an MV-chain and $a, b \in M$, then $a \vee b = max(a, b)$ and $a \wedge b = min(a, b)$, whence clearly the distributive laws hold in M.

[2]For a detail exposition of Boolean algebras theory see e.g.[12], [16], [18], [19], [20], [21].

Using the above join and meet formulas, the distributive laws can bie equivalently reformulated as MV-equations; since these equations are satisfied by all MV-chains, by Corollary 2.1.4.7 we obtain

Proposition 2.1.5.1 For any MV-algebra M, $L(M)$ is a distributive lattice with smallest element 0 and greatest element 1.

Definition 2.1.5.2 An element x of a lattice L with 0 and 1 is said to be *complemented* iff there is an element $y \in L$ (the complement of x) such that $x \vee y = 1$ and $x \wedge y = 0$. When L is distributive each $z \in L$ has al most one complement, denoted z^c. We further let

$$B(L)$$

be the set of all complemented elements of the distributive lattice L. Note that 0 and 1 are elements of $B(L)$, because $0^c = 1$ and $1^c = 0$. As a matter of fact, $B(L)$ is a sublattice of L wich is also a Boolean algebra. For any MV-algebra M we shall write $B(M)$ as an abbreviation of $B(L(M))$. Elements of $B(M)$ are called the Boolean elements of M.

Theorem 2.1.5.3 For every element x in an MV-algebra M the following conditions are equivalent:

(i) $x \in B(M)$.

(ii) $x \vee x^* = 1$.

(iii) $x \wedge x^* = 0$.

(iv) $x + x = x$.

(v) $x \circ x = x$.

(vi) $x + y = x \vee y$, for all $y \in M$.

(vii) $x \circ y = x \wedge y$, for all $y \in M$.

Proof. The following equivalences are trivial: $(ii) \Leftrightarrow (iii)$, $(iv) \Leftrightarrow (v)$,

$(vi) \Leftrightarrow (vii)$. It is also trivial that $(vi) \Rightarrow (iv)$. Further, the equivalent conditions (ii) and (iii) state that x^* is the complement of x. Thus, in particular $(iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: By elementary manipulations, using Lemma 2.1.1.8 and Proposition 2.1.1.11 we have

$$x^* = x^* + 0 = x^* + (x \wedge x^c) = (x^* + x) \wedge (x^* + x^c) = x^* + x^c.$$

Thus,

$$x^c \leq x^*$$

and

$$1 = x \vee x^c \leq x \vee x^* \leq 1,$$

and we are done.

$(iii) \Rightarrow (vi)$: Using Proposition 2.1.2.3, together with the Subdirect Representation Theorem 2.1.3.3 and the inequality $x \vee y \leq x + y$, (which also is an immediate consequence of Theorem 2.1.3.3) we have

$$d(x + y, x \vee y) = (x + y) \circ (x \vee y)^*$$

$$= (x + y) \circ (x^* \wedge y^*) \leq ((x + y) \circ x^*) \wedge ((x + y) \circ y^*)$$

$$= x^* \wedge y \wedge y^* \wedge x.$$

Therefore, $x \wedge x^* = 0$ implies $d(x + y, x \vee y) = 0$, whence $x + y = x \vee y$.

$(iv) \Rightarrow (ii)$: By hypothesis,

$$1 = x^* + x = (x + x)^* + x = x^* \vee x.$$

□

Corollary 2.1.5.4 $B(M)$ is a subalgebra of the MV-algebra M. A subalgebra B of M is a Boolean algebra iff $B \subseteq B(M)$.

Corollary 2.1.5.5 An MV-algebra M is a Boolean algebra if and only if the operation $+$ is idempotent, i.e., the equation $x + x = x$ is satisfied by M.

2.2 MV and MV-effect algebras

Definition 2.2.1 An *MV-effect algebra* is a lattice ordered effect algebra E in which, for all $a, b \in E$, $(a \vee b) \ominus b = a \ominus (a \wedge b)$.

Lemma 2.2.2 Let E be an effect algebra and $a, b, c \in E$ then:

(i) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

(ii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

(iii) If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(iv) If $b \leq c$ and $a \leq c \ominus b$, then $b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.

Proof. (i) and (ii) follow directly from Definition 1.1.9.

(iii) From (ii) and Definition 1.1.9 we get that

$$(c \ominus a) \ominus (c \ominus b) = b \ominus a \leq c \ominus a$$

and by (i),

$$(c \ominus a) \ominus (b \ominus a) = (c \ominus a) \ominus ((c \ominus a) \ominus (c \ominus b)) = c \ominus b$$

(iv) From the hypotheses it follows that $a \leq c \ominus b \leq c$, and from (ii) we obtain

$$c \ominus (c \ominus b) \leq c \ominus a, i.e., b \leq c \ominus a$$

Since by (iii),

$$(c \ominus b) \ominus a \leq c \ominus a,$$

we get from (iii):

$$(c \ominus a) \ominus ((c \ominus b) \ominus a) = c \ominus (c \ominus b) = b,$$

therefore,

$$(c \ominus a) \ominus b = (c \ominus a) \ominus ((c \ominus a) \ominus ((c \ominus b) \ominus a)) = (c \ominus b) \ominus a$$

□

Lemma 2.2.3 Let E be a lattice ordered effect algebra, $a, b, c \in E$, $a \leq c, b \leq c$. Then

$$c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$$

Proof. From the inequalities $a \wedge b \leq a \leq c$ and $a \wedge b \leq b \leq c$ it follows that

$$c \ominus a \leq c \ominus (a \wedge b)$$

and

$$c \ominus b \leq c \ominus (a \wedge b).$$

For $\omega \in E$ with $c \ominus a \leq \omega, c \ominus b \leq \omega$, then

$$c \ominus a = (c \ominus a) \wedge c \leq \omega \wedge c \leq c \leq c,$$

which gives,

$$c \ominus (\omega \wedge c) \leq a,$$

and similarly,

$$c \ominus (\omega \wedge c) \leq b,$$

therefore,

$$c \ominus (\omega \wedge c) \leq a \wedge b.$$

Then we obtain:

$$c \ominus (a \wedge b) \leq \omega \wedge c \leq \omega$$

wich implies that:

$$c \ominus (a \wedge b)$$

is the least upper bound of the set $\{\mathrm{c}\ominus a, c \ominus b\}$. □

Proposition 2.2.4 Let E an MV-effect algebra. We define a binary operation $-$ by $b - a := b \ominus (a \wedge b)$, then $(a - b) - c = (a - c) - b$ for every $a, b, c \in E$.

Proof. If $c \leq a$, by Part(iv) Lemma 2.2.2,Lemma 2.2.3 and Definition 2.2.1,

$$\begin{aligned}(a-b)-c &= (a \ominus (a \wedge b)) \ominus ((a \ominus (a \wedge b)) \wedge c) \\ &= (a \ominus ((a \ominus (a \wedge b)) \wedge c) \ominus (a \wedge b) \\ &= ((a \ominus (a \ominus (a \wedge b))) \vee (a \ominus c)) \ominus (a \wedge b) \\ &= ((a \wedge b) \vee (a \ominus c)) \ominus (a \wedge b) \\ &= (a \ominus c) \ominus ((a \ominus c) \wedge (a \wedge b)) \\ &= (a \ominus c) \ominus ((a \ominus c) \wedge b) \\ &= (a \ominus (a \wedge c)) \ominus ((a \ominus (a \wedge c)) \wedge b) = (a-c)-b\end{aligned}$$

Let $a, b, c \in E$, then:

$$(a-b)-(c \wedge a) = (a-(a \wedge c))-b,$$

but,

$$(a-b)-(c \wedge a) = (a-b)-c$$

and

$$(a-(a \wedge c))-b = (a-c)-b.$$

Hence,

$$(a-b)-c = (a-c)-b.\square$$

Theorem 2.2.5 *Every MV-effect algebra is an MV-algebra.*

Proof. Let E be an MV-effect algebra. Let us define $a^* := 1-a = 1 \ominus a$ and $a+b := (a^*-b)^*$. We have to check the MV-algebra axioms:

(MV4) and (MV8) are immediate.

(MV2) By Proposition 2.2.4,

$$(a+b)^* = a^*-b = (1-a)-b$$

$$= (1-b)-a = b^*-a = (b+a)^*.$$

Then:

$$a+b = b+a.$$

(MV1) By Proposition 2.2.4,

$$((a+b)+c)^* = (a+b)^* - c = (a^* - b) - c$$

$$= (a^* - c) - b = (a+c)^* - b = ((a+c)+b)^*.$$

Then, by (MV2):

$$(a+b)+c = a+(b+c).$$

(MV3) $a+0 = (a^* - 0)^* = (a^* \ominus 0)^* = (a^*)^* = a$.

(MV5) $a+1 = (a^* - 1)^* = 0^* = 1$.

(MV6) It is immediate that $b-(b-a) = a \wedge b = a-(a-b)$. Then:

$$a+(a+b^*)^* = (a^* - (a+b^*)^*)^* = (a^* - ((a^* - b^*)^*)^*)^* = (a^* - (a^* - b^*))^*$$

$$= (b^* - (b^* - a^*))^* = (b^* - (b+a^*)^*)^* = b+(b+a^*)^*.$$

(MV7) $a+a^* = (a^* - a^*)^* = (a^* \ominus a^*)^* = 0^* = 1$.

□

2.3 Bibliographical remarks

In the early twenties Łukasiewicz introduced a propositional calculus in which the propositions may have a truth value any real number between 0 and 1. The basic connectives of this calculus were *implication* $\Rightarrow$ and *negation* $\sim$ having as "truth-tables" equations $x \Rightarrow y = min(1, 1-x+y)$ and $\sim x = x \Rightarrow 0 = 1-x$ for each $x, y \in [0,1]$, respectively.
Moreover, Łukasiewicz conjectured that all tautologies for the calculus can be derived from the following axiom-schemes, using as unique deduction rule *modus ponens*:

(L1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$

(L2) $(\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma))$

(L3) $(\sim \alpha \Rightarrow \sim \beta) \Rightarrow (\beta \Rightarrow \alpha)$

(L4) $((\alpha \Rightarrow \beta) \Rightarrow (\beta \Rightarrow \alpha)) \Rightarrow (\beta \Rightarrow \alpha)$

MV-algebras were originally introduced by Chang in [4] with the aim of given an algebraic proof of Łukasiewicz conjecture.

The results of the first section of this chapter are borrowed from [5].

As reference for the second section of this chapter we mention the book [9] and the papers [6] and [7].

Chapter 3

MV-Pairs and MV-Algebras

Let B be a Boolean algebra. We write $\mathrm{Aut}(B)$ for the group of all automorphisms of B. Let G be a subgroup of $\mathrm{Aut}(B)$. For $a, b \in B$, we write $a \sim_G b$ iff there exists $f \in G$ such that $b = f(a)$. Obviously, $\sim_G$ in an equivalence relation. We write $[a]_G$ for the equivalence class of an element a of B. A pair (B, G), where B is a Boolean algebra and G is a subgroup of $\mathrm{Aut}(B)$ is called a *BG-pair*.

Let $(P, \leq)$ be a poset. Let us write,

$$max(P) = \{m \in P : m \leq x \Rightarrow x = m\}$$

that means, max(P) is the set of all maximals elements of the poset P.

Let B a Boolean algebra, let G be a subgroup ol $\mathrm{Aut}(B)$. For all $a, b \in B$, we write

$$L(a, b) = \{a \wedge f(b) : f \in G\}$$

and

$$L^+(a, b) = \{g(a) \wedge f(b) : f, g \in G\}$$

Note that $L(a, b) \subseteq L^+(a, b)$.

Definition 3.1 Let B be a Boolean algebra, let G be a subgroup of $\mathrm{Aut}(B)$. We say that $(B; G)$ is an *MV-pair* iff the following two conditions are satisfied.

(MVP1) For all $a, b \in B$, $f \in G$ such that $a \leq b$ and $f(a) \leq b$, there is $h \in G$ such that $h(a) = f(a)$ and $h(b) = b$.

(MVP2) For all $a, b \in B$ and $x \in L(a, b)$, there exist $m \in max(L(a, b))$ with m$\geq x$.

Example 3.2 For every finite Boolean algebra B, $(B, Aut(B))$ is an MV-pair.[1] Let

$$At(B) = \{a_1, a_2, ..., a_n\}$$

the set of atoms of B, and $\Lambda = \{\sigma \in At(B)^{At(B)} : \sigma \text{ is a bijection}\}$. For every bijection $\sigma \in \Lambda$, we can define $f_\sigma \in Aut(B)$, associated with the permutation σ by:

$$f_\sigma(\bigvee a_{i_j}) = \bigvee \sigma(a_{i_j})$$

It is clear that $Aut(B) = \{f_\sigma\}_{\sigma \in \Lambda}$. Let us see that $(B, Aut(B))$ is an MV-pair. Let us prove (MVP1) $A \subseteq At(B)$, $C \subseteq At(B)$, $f \in Aut(B)$ such that $\bigvee A \leq \bigvee C$ and $f(\bigvee A) \leq \bigvee C$. We write $f(A) = \{f(a) : a \in A\}$, since

$$Card(A) = Card(f(A))$$

and

$$card(f(A) - A) = card(A - f(A)),$$

hence there exist the following bijections

$$\sigma_1 : A \to f(A)$$

and

$$\sigma_2 : (f(A) - A) \to (A - f(A))$$

Now we define $\sigma : At(B) \to At(B)$ as follows

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if} \quad x \in A \\ \sigma_2(x) & \text{if} \quad x \in f(A) - A \\ x & \text{if} \quad x \notin A \cup (f(A) - A) \end{cases}$$

[1] Recall that every finite Boolean algebra B is atomic

Then, it is clear that $f_\sigma(\bigvee A) = f(\bigvee A)$ and $f_\sigma(\bigvee C) = \bigvee C$. This concludes the proof of (MVP1).
Now, let us prove condition (MVP2). Let $A \subseteq At(B)$, $C \subseteq At(B)$ and $X = (\bigvee A) \wedge f(\bigvee C)$, with $f \in Aut(B)$. Suppose first that $card(A) \leq card(C)$, then there are $S \subseteq f(C)$ and a bijection

$$\sigma_3 : A \to S.$$

Since

$$A = (A - S) \cup (A \cap S)$$

and

$$S = (S - A) \cup (A \cap S)$$

then

$$card(A - S) = card(S - A),$$

then there is a bijection

$$\sigma_4 : (A - S) \to (S - A)$$

Then, we define $\sigma : At(B) \to At(B)$ as follows

$$\sigma(x) = \begin{cases} \sigma_3^{-1}(x) & \text{if} \quad x \in S \\ \sigma_4(x) & \text{if} \quad x \in A - S \\ x & \text{if} \quad x \notin S \cup (A - S) \end{cases}$$

and we obtain the maximal element $m = \bigvee A \wedge f_\sigma(f(\bigvee C)) = \bigvee A$ such that $m \geq X$.
Suppose now that $card(A) > card(C)$. Since

$$A = (A - f(C)) \cup (A \cap f(C))$$

and

$$f(C) = (f(C) - A) \cup (A \cap f(C)),$$

then

$$card(A - f(C)) > card(f(C) - A).$$

then there are $S \subseteq A - f(C)$ and a bijection

$$\sigma_5 : (f(C) - A) \to S$$

Now we consider the next permutation $\sigma : At(B) \to At(B)$

$$\sigma(x) = \begin{cases} \sigma_5(x) & \text{if} \quad x \in f(C) - A \\ \\ \sigma_5^{-1}(x) & \text{if} \quad x \in S \\ \\ x & \text{if} \quad x \notin S \cup (f(C) - A) \end{cases}$$

Then $m = \bigvee A \wedge f_\sigma(f(\bigvee C))$ is the searched maximal element.

Example 3.3 Let B a finite Boolean algebra with three atoms a_1, a_2, a_3. The mapping f given by

x	0	a_1	a_2	a_3	a_1^c	a_2^c	a_3^c	1
$f(x)$	0	a_2	a_3	a_1	a_2^c	a_3^c	a_1^c	1

defines an automorphism of B and $G = \{id, f, f^2\}$ is a subgroup of $Aut(B)$. However, (B, G) is not an MV-pair. Indeed, we have $a_1 \leq a_3^c$ and $f(a_1) = a_2 \leq a_3^c$, but there is not $h \in G$ such that $h(a_1) = f(a_1)$ and $h(a_3^c) = a_3^c$.

Example 3.4 Let $2^{\mathbb{Z}}$ be the Boolean algebra of all subsets of $\mathbb{Z}$. Then $(2^{\mathbb{Z}}, Aut(2^{\mathbb{Z}}))$ is not an MV-pair. Indeed, let $f \in Aut(2^{\mathbb{Z}})$ be the automorphism of $2^{\mathbb{Z}}$ associated with the permutation $f(n) = n + 1$. Let $A = B = \mathbb{N}$. We see that $f(A) = A \setminus \{0\}$, $A \subseteq B$ and $f(A) \subseteq B$. However, there is no $h \in Aut(2^{\mathbb{Z}})$ such that $h(A) = f(A)$ and $h(B) = B$, simply because $A = B$ implies that $h(A) = h(B)$, but $f(A) \neq B$.

Example 3.5 Let $\mathcal{A}$ be the Boolean algebra of all those subsets of $\mathbb{N}$ that are either finite or cofinite, and $\Upsilon = \{f \in \mathbb{N}^{\mathbb{N}} : f \text{ is a bijection}\}$. For every bijection $f : \mathbb{N} \to \mathbb{N}$, let ψ_f be the mapping $\psi_f : \mathcal{A} \to \mathcal{A}$ given by

$$\psi_f(\bigcup_{j \in J} n_j) = \bigcup_{j \in J} f(n_j)$$

and let

$$G = \{\psi_f\}_{f\in\Upsilon}.$$

First,we will prove that $G = Aut(\mathcal{A})$. Trivially, $Aut(\mathcal{A}) \subseteq G$. To prove the converse inclusion, let $\phi \in Aut(\mathcal{A})$ and $A \in \mathcal{A}$, $A = \bigcup_{i\in I}\{n_i\}$. Since $\{n_i\} \subseteq A$ for every $i \in I$, then[2] $\phi(n_i) \in \phi(A)$ for every $i \in I$, and $\bigcup_{i\in I}\phi(n_i) \subseteq \phi(A)$. Suppose now that there is $B \in \mathcal{A}$ such that $\phi(n_i) \in B$ for every $i \in I$, hence $n_i \in \phi^{-1}(B)$. Then $A = \bigcup_{i\in I} n_i \subseteq \phi^{-1}(B)$, therefore $\phi(A) \subseteq B$, and we conclude that $\phi(A) = \bigcup_{i\in I}\phi(n_i)$. This yields the desired inclusion.

Let us see that $(\mathcal{A}, G)$ is an MV-pair. Let $A, B \in \mathcal{A}$, $\psi_f \in G$ such that $A \subseteq B$, $f(A) \subseteq B$. Suppose that A is finite, since

$$card(A) = card(A - f(A)) + card(A \cap f(A))$$

and

$$card(f(A)) = card(f(A) - A) + card(A \cap f(A))$$

then

$$card(A - f(A)) = card(f(A) - A).$$

Then we can take a bijection $g : (A - f(A)) \to (f(A) - A)$, and now we can define $h : \mathbb{N} \to \mathbb{N}$ as follows:

$$h(x) = \begin{cases} g(x) & \text{if} \quad x \in A - f(A) \\ g^{-1}(x) & \text{if} \quad x \in f(A) - A \\ x & \text{if} \quad x \notin A - f(A) \cup f(A) - A \end{cases}$$

Then $h(A) = f(A)$ and $h(B) = B$.
Suppose now that A is infinite, so $card(A^c) < \infty$. Since

$$card(A^c) = card(f(A) - A) + card(A^c \cap f(A)^c),$$

and

$$card(f(A)^c) = card(A - f(A)) + card(A^c \cap f(A)^c).$$

Then

$$card(f(A) - A) = card(A - f(A)) < \infty.$$

[2] In every Boolean algebra, we write $p \leq q$ in case $p \wedge q = p$, or, equivalently, $p \vee q = q$.

Let $h : \mathbb{N} \to \mathbb{N}$ the same function defined previously, so $h(A) = f(A)$ and $h(B) = B$.
It remains to show condition (MVP2). Let $A, B \in \mathcal{A}$ and $X \in L(A, B)$, $X = A \cap f(B)$, we argue by cases as follows:

Case 1: $card(A) < \infty$ and $card(A) \leq card(B)$. Let $S \subseteq f(B)$ and a bijection $g_1 : S \to A$. Now we consider the function $h_1 : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$h_1(x) = \begin{cases} g_1(x) & \text{if} \quad x \in S \\ g_1^{-1}(x) & \text{if} \quad x \in A - S \\ x & \text{if} \quad x \notin S \cup (A - S) \end{cases}$$

Then, $M = A \cap h_1(f(B)) = A$ is an element of $L(A, B)$ such that $M \supseteq X$.

Case 2: $card(B) < \infty$ and $Card(B) < Card(A)$. Since

$$A = (A - f(B)) \cup (A \cap f(B))$$

and,

$$f(B) = (f(B) - A) \cup (A \cap f(B)).$$

Then, there are $S \subseteq A - f(B)$ and a bijection $g_2 : f(B) - A \to S$. Now we define the function $h_2 : \mathbb{N} \to \mathbb{N}$:

$$h_2(x) = \begin{cases} g_2(x) & \text{if} \quad x \in f(B) - A \\ g_2^{-1}(x) & \text{if} \quad x \in S \\ x & \text{if} \quad x \notin S \cup (f(B) - A) \end{cases}$$

Then, $M = A \cap h_2(f(B)) = A \cap (f(B) \cup S)$ is the searched maximal element.

There remains to consider

Case 3: $Card(A) = Card(B) = \aleph_0$. Since $f(B) - A \subseteq A^c$ and $A - f(B) \subseteq f(B)^c$, $f(B) - A$ and $A - f(B)$ are finites.
First, we suppose that $card(A - f(B)) \leq card(f(B) - A)$, then there are

$S \subseteq f(B) - A$ and a bijection $g_3 : S \to A - f(B)$. If we consider the function $h_3 : \mathbb{N} \to \mathbb{N}$ defined by:

$$h_3(x) = \begin{cases} g_3(x) & \text{if} \quad x \in S \\ g_3^{-1}(x) & \text{if} \quad x \in A - f(B) \\ x & \text{if} \quad x \notin S \cup (A - f(B)) \end{cases}$$

we obtain the maximal element $M = A \cap h_3(f(B)) = A$, that verifies the condition (MVP2).
Finally, if $card(A - f(B)) > card(f(B) - A)$, there are $S \subseteq A - f(B)$ and a bijection $g_4 : f(B) - A \to S$. Now we define the function $h_4 : \mathbb{N} \to \mathbb{N}$

$$h_4(x) = \begin{cases} g_4(x) & \text{if} \quad x \in f(B) - A \\ g_4^{-1}(x) & \text{if} \quad x \in S \\ x & \text{if} \quad x \notin S \cup (f(B) - A) \end{cases}$$

Then we obtain $M = A \cap h_4(f(B)) = (A \cap f(B)) \cup (A \cap S)$ that satisfies the condition (MVP2).

Lemma 3.6 Let be a Boolean algebra, let G be a subgroup of $\mathrm{Aut}(B)$. Then the following conditions are equivalent.

(i) (MVP1).

(ii) For all $a, b \in B$, $f \in G$ such that $a \leq b$ and $a \leq f(b)$, there is $h \in G$ such that $h(b) = f(b)$ and $h(a) = a$.

(iii) For all $a, b \in B$, $f \in G$ such that $a \wedge b = 0$ and $a \wedge f(b) = 0$, there is $h \in G$ such that $h(b) = f(b)$ and $h(a) = a$.

Proof. $(i) \Rightarrow (ii)$ Let $a, b \in B$, $f \in G$ such that $a \leq b$ and $a \leq f(b)$, then $b^c \leq a^c$ and $f(b^c) \leq a^c$. By (i) there is $h \in G$ such that $h(b^c) = f(b^c)$ and $h(a^c) = a^c$, so $h(b) = f(b)$ and $h(a) = a$.

$(ii) \Rightarrow (iii)$ Let $a, b \in B$, $f \in G$ such that $a \wedge b = 0$ and $a \wedge f(b) = 0$, then

$a \leq b^c$ and $a \leq f(b^c)$. By (ii) there is $h \in G$ such that $h(b^c) = f(b^c)$ and $h(a) = a$.

$(iii) \Rightarrow (i)$ Let $a, b \in B$, $f \in G$ such that $a \leq b$ and $f(a) \leq b$, then:

$$b^c \wedge (b \wedge a) = 0, f(a) \wedge b^c = 0$$

and

$$b^c \wedge f(b \wedge a) = 0.$$

By (iii) there exists $h \in G$ such that $h(a) = f(a)$ and $h(b^c) = b^c$. This complete the proof. □

Lemma 3.7 Let (B, G) be an MV-pair, let $a, b \in B$ and let m be a maximal element of $L(a, b)$. For all $f \in G$, $f(m)$ is a maximal element of $L^+(a, b)$.

Proof. Suppose that there is some element $y \in L^+(a, b)$ with $y \geq f(m)$ an write $y = g_1(a) \wedge f_1(b)$, where $g_1, f_1 \in G$. Since $m \in L(a, b)$, $a \geq m$ and since

$$a \wedge g_1^{-1}(f_1(b)) = g_1^{-1}(g_a \wedge f_1(b))$$

$$= g_1^{-1}(y) \geq g_1^{-1}(f(m)) = (g_1^{-1} \circ f)(m),$$

we see that $a \geq (g_1^{-1} \circ f)(m)$.

By (MVP1), $a \geq (g_1^{-1} \circ f)(m)$ and $a \geq m$ imply that there exists $h \in G$ such that $h(a) = a$ and $h(m) = (g_1^{-1} \circ f)(m)$. We apply h^{-1} to both sides of inequality

$$a \wedge g_1^{-1}(f_1(b)) \geq (g_1^{-1} \circ f)(m)$$

to obtain

$$h^{-1}(a \wedge g_1^{-1}(f_1(b))) =$$

$$a \wedge h^{-1}(g_1^{-1}(f_1(b))) \geq h^{-1}((g_1^{-1} \circ f)(m)) = m.$$

Since m is a maximal element of $L(a, b)$, $a \wedge h^{-1}(g_1^{-1}(f_1(b))) \geq m$ implies that:

$$a \wedge h^{-1}(g_1^{-1}(f_1(b))) = m.$$

After we apply the mapping $g_1 \circ h$ on both sides of the latter equality we obtain:

$$y = g_1(a) \wedge f_1(b) = f(m).$$

Thus, $f(m)$ is maximal in $L^+(a,b)$. $\square$

Recall that a Boolean algebra $(B;\leq,0,1,\wedge,\vee)$,regarded as a bounded distributive lattice, can be organized into an effect algebra $(E;\oplus,0,1)$, if the partial binary operation $\oplus$ is defined by $p\oplus q = p\vee q$ iff $p\wedge q = 0$, in this case we denote $p\oplus q = p\dot{\vee}q$.

Theorem 3.8 *Let (B,G) be an MV-pair. Then $\sim_G$ is an effect algebra congruence on B and $B/\sim_G$ is an MV- effect algebra.*

Proof. Let B be a Boolean Algebra, recall that B can be organized into an effect algebra.[3]

We shall prove that $\sim_G$ is an effect algebra congruence on B.

Obviously, $\sim_G$ is an equivalence relation.

To prove (C2), Let $a_1, a_2, b_1, b_2 \in B$ be such that $a_1 \sim_G a_2$, $b_1 \sim_G b_2$ and $a_1\dot{\vee}b_1$, $a_2\dot{\vee}b_2$ exist. There are $f_a, f_b \in G$ such that $f_a(a_1) = a_2$ and $f_b(b_1) = b_2$.

We see that $b_2^c \geq a_2$ and that implies

$$b_1^c = f_b^{-1}(b_2^c) \geq f_b^{-1}(a_2) = f_b^{-1}(f_a(a_1)) = (f_b^{-1}\circ f_a)(a_1)$$

By (MVP1), $a_1 \leq b_1^c$ and $(f_b^{-1}\circ f_a)(a_1) \leq b_1^c$ imply that there is $h\in G$ such that

$$h(a_1) = (f_b^{-1}\circ f_a)(a_1)$$

and

$$h(b_1^c) = b_1^c.$$

Therefore,

$$f_b(h(a_1\dot{\vee}b_1)) = f_b(h(a_1)\dot{\vee}h(b_1)) =$$

$$f_b((f_b^{-1}\circ f_a)(a_1)\dot{\vee}b_1) = f_a(a_1)\dot{\vee}f_b(b_1) = a_2\dot{\vee}b_2$$

and

$$a_1\dot{\vee}b_1 \sim_G a_2\dot{\vee}b_2.$$

[3]cf. Example 1.1.2.

Let us prove (C3). Let $a_1, a_2 \in B$ such that $a_1 \dot{\vee} a_2$ exists and $a_1 \dot{\vee} a_2 \sim_G b$. Then There is $f \in G$ such that $f(a_1 \dot{\vee} a_2) = b$ and we may put $b_1 = f(a_1)$ and $b_2 = f(a_2)$.

It is easy to see that $\sim_G$ preserves ^{c} operation, so (C4) is satisfied.

By Theorem 1.1.15, since $\sim_G$ is an effect algebra congruence, $B/_{\sim_G}$ is an effect algebra. By (iii) and (vii) of Theorem 1.3.9 B satisfies the Riesz descomposition property, then by Proposition 1.2.6 $B/_{\sim_G}$ satisfies the Riesz descomposition property, and by (ii) of Theorem 1.3.9, for $a, b, c \in B$, $B/\sim_G$ satisfies $[a] \ominus ([a] \wedge [b]) = ([a] \wedge [b]) \ominus [b]$.

It remains to prove that $B/_{\sim_G}$ is a lattice[4] . By Proposition 1.1.11 an effect algebra is a lattice iff it is a (join or meet) semilattice, it suffices to prove that for all $a, b \in B$, $[a]_G \wedge [b]_G$ exists in $B/_{\sim_G}$.

Let $a, b \in B$, we shall prove that every common lower bound of $[a]_G, [b]_G$ is under a maximal common lower bound of $[a]_G, [b]_G$. If $[c]_G \leq [a]_G, [b]_G$ then, by Lemma 1.1.14, there is $c_1 \sim_G c$ such that $c_1 \leq a$ and, again by lemma 1.1.14, $b_1 \sim_G b$ such that $c_1 \leq b_1$. As $b_1 \sim_G b$, there is $f \in G$ such that $b_1 = f(b)$. Thus,

$$c \sim_G c_1 \leq a \wedge f(b) \in L(a, b).$$

By (MPV2), there is $m \in max(L(a, b))$ with $a \wedge f(b) \leq m$. Obviously, $m \in L(a, b)$ implies that $[m]_G \leq [a]_G, [b]_G$. Therefore, for every common lower bound $[c]_G$ of $[a]_G, [b]_G$, there is $m \in max(L(a, b))$ such that

$$[c]_G \leq [m]_G \leq [a]_G, [b]_G.$$

Let us prove that $[m]_G$ is a maximal common lower bound of $[a]_G, [b]_G$ in $B/_{\sim_G}$. Suppose that

$$[m]_G \leq [x]_G \leq [a]_G, [b]_G.$$

By Lemma 1.1.14, there are $m_1 \sim_G m$, $x_1 \sim_G x$ and $b_1 \sim_G b$ such that

$$m_1 \leq x_1 \leq a, b_1.$$

There is $f \in G$ such that $b_1 = f(b)$. We see that:

$$x_1 \leq a \wedge f(b) \in L(a, b) \subseteq L^+(a, b).$$

[4]If $B/_{\sim_G}$ is finite , by Proposition 1.2.5 and Lemma 1.2.4 $B/_{\sim_G}$ is a lattice.

There is $g \in G$ such that $m_1 = g(m)$. By Lemma 3.7, $m_1 = g(m)$ is a maximal element of $L^+(a,b)$. Therefore, $m_1 = a \wedge f(b)$ and hence $x_1 = m_1$. This implies $[m]_G = [x]_G$.

Let $[m_1]_G, [m_2]_G$ be a maximal common lower bounds of $[a]_G, [b]_G$. Since $B/_{\sim_G}$ satisfies the Riesz descomposition property, by Proposition 1.2.5 $B/_{\sim_G}$ satisfies the Riesz interpolation property. By the Riesz interpolation property, there is $[m]_G$ such that $[m_1]_G, [m_2]_G \leq [m]_G \leq [a]_G, [b]_G$. Since $[m_1]_G, [m_2]_G$ are maximal, $[m_1]_G = [m]_G = [m_2]_G$. Since every common lower bound of $[a]_G, [b]_G$ is under a maximal one, an there is a single maximal common lower bound of $[a]_G, [b]_G, [a]_G \wedge [b]_G$ exists, and this completes the proof of the theorem. □

Corollary 3.9 $B/\sim_G$ is an MV-algebra.

Proof. Follows from Theorem 2.2.5. □

Corollary 3.10 $B/\sim_G$ is a Boolean algebra iff for every $a \in B$, $[a] + [a] = [a]$.

Proof. Follows from Corollary 2.1.5.5.

Corollary 3.11 Let B a finite Boolean algebra and G be a subgroup of $Aut(B)$. Then the following conditions are equivalent.

(i) $B/\sim_G$ is a Boolean algebra.

(ii) For every $a \in B$, $[a] + [a] = [a]$.

(iii) If $a \in G$, then $[a] \wedge [a]^* = 0$.[5]

(iv) $G = \{id\}$.

Proof. $(iv) \Rightarrow (i)$: is trivial.

[5]An effect algebra E that satisfies $p \wedge p' = 0$ for every $p \in E$, is an orthoalgebra. An *orthoalgebra* is an effect algebra in which the zero-one law is replaced by the stronger (Consistency law) $p \perp p \Rightarrow p = 0$, see [14].

$(i) \Leftrightarrow (ii)$: Follows from Corollary 3.10.

$(ii) \Rightarrow (iii)$: Follows from Theorem 2.1.5.3.

$(iii) \Rightarrow (iv)$: Let $a_1, ..., a_n$ the atoms of B. Suppose, without any loss of generality, that there exists $f \in Aut(B)$ such that $f(a_1) = a_j$ for $2 \leq j \leq n$. Then:

$$f(a_1) \wedge a_1^* = a_j \wedge (a_2 \vee a_3 \vee ... \vee a_n) = a_j$$

and $a_j \in L(a_1, a_1^*)$. By proof of Theorem 3.8 we know that there is $m \in max(L(a_1, a_1^*))$ such that $[a_1] \wedge [a_1]^* = [m]$. By hypothesis $m = 0$ then $a_j = 0$, a contradiction. □

Example 3.12 Let B be the finite Boolean algebra with n atoms $a_1, a_2, ..., a_n$, and let G be the group of all automorphisms of B. It is clear that

$$B/\sim_G = \{[0], [a_1], [a_1 \vee a_2], ..., [a_1 \vee ... \vee a_n] = [1]\},$$

where: $[0] = \{0\}$, $[a_1] = At(B)$, $[a_1 \vee a_2] = \{a_{i_j} \vee a_{i_k} : i_j \neq i_k, 1 \leq i_j, i_k \leq n\}$,..., $[a_1 \vee a_2 ... \vee a_n] = \{a_1 \vee a_2 \vee ... \vee a_n\}$.

Let $r, l \leq n$ we define a binary operation $+$ and * as follows:

$$[a_1 \vee ... \vee a_l] + [a_1 \vee ... \vee a_r] := \begin{cases} [a_1 \vee ... \vee a_{r+l}], & \text{if} \quad r + l < n \\ [1], & \text{if} \quad r + l \geq n \end{cases}$$

$$[0] + [a_1 \vee ... \vee a_r] := [a_1 \vee ... \vee a_r].$$

$$[a_1 \vee ... \vee a_l]^* := [(a_1 \vee ... \vee a_l)^c]$$

It is not difficult to see that $(B/\sim_G, +, *, [0], [a_1 \vee ... \vee a_n])$ is an MV-algebra isomorphic to L_{n+1}[6].

Example 3.13 Let $(\mathcal{A}, G)$ be the MV-pair of the Example 3.6. Let $A, B \in \mathcal{A}$, $A \sim_G B$ iff $card(A) = card(B)$.
Then:

$\mathcal{A}/\sim_G = \{[\emptyset], [\{1\}], [\{1,2\}], [\{1,2,3\}], ...\} \cup \{[\mathbb{N}], [\{1\}^c], [\{1,2\}^c], [\{1,2,3\}^c], ...\}$,

[6]cf. examples 2.1.1.3 and 2.1.1.5

where

$$[\emptyset] = \emptyset,$$
$$[\{1\}] = \{S \subseteq \mathbb{N} : card(S) = 1\},$$
$$[\{1,2\}] = \{S \subseteq \mathbb{N} : card(S) = 2\}, ...$$
$$[\mathbb{N}] = \mathbb{N},$$
$$[\{1^c] = \{S \subseteq \mathbb{N} : card(S^c) = 1\},$$
$$[\{1,2^c] = \{S \subseteq \mathbb{N} : card(S^c) = 2\}, ...$$

The zero element of $\mathcal{A}/\sim_G$ is $[\emptyset]$ and the + and * are defined as follows: If $card(A) = r$ and $card(B) = k$:

$$[A] + [B] := [\{1, 2, ..., r + l\}]$$

If $card(A), card(B^c)$ are finites and $card(A) - card(B^c) < 0$:

$$[A] + [B] := [\{1, 2, ..., -(card(A) - card(B^c)\}^c],$$

In any other case:

$$[A] + [B] := [\mathbb{N}].$$

And:

$$[A]^* := [A^c].$$

$\mathcal{A}/\sim_G$ is isomorphic to the MV-algebra:

$$\Sigma(\mathbb{Z}) = \{(0, a) : a \in \mathbb{Z}^+\} \cup \{(1, b) : b \in \mathbb{Z}^-\}.^7$$

The proof of this fact is a bit longer, but straightforward.
The zero element of $\Sigma(\mathbb{Z})$ is $(0, 0)$ and the operations + and $*$ are defined as follows:

$$(i, a) + (j, b) := \begin{cases} (0, a + b) & \text{if} \quad i + j = 0 \\ (1, (a + b) \wedge 0) & \text{if} \quad i + j = 1 \\ (1, 0) & \text{if} \quad i + j = 2 \end{cases}$$

and:

$$(i, a)^* = (1 - i, -a).$$

[7] The MV-algebra $\Sigma(\mathbb{Z})$ is the first example of a nonsemisimple MV-algebra, see [5].

3.1 Bibliographical remarks

As a reference for this chapter, we mention the paper [17].

Bibliography

[1] **Bennet M K and Foulis D J** (1995), *Phi-Symmetric Effect Algebras.* Foundations of Physics, 25:1699-1722.

[2] **Birkhoff G**, *Lattice Theory*, Amer. Math. Soc. Coll. Publ., Vol. 25, Providence, Rhode Island (1967).

[3] **Boole G**, *An investigation of the Laws of Thought*, Macmillan (1854), reprinted by Dover Press, Nex York (1967).

[4] **Chang C C** (1958), *Algebraic analysis of many-valued logics.* Transactions of the American Mathematical Society, 88: 467-490.

[5] **Cignoli R, D'Ottaviano I and Mundici D**, *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Academic Publishers, Dordrecht (2000).

[6] **Chovanec F and Kôpka F** (1995), *D-Lattices.* International Journal of Theoretical Physics, 34: 1297-1302.

[7] **Chovanec F and Kôpka F** (1997), *Boolean D-Posets.* Tatra Mountains Mathematical Publications, 10: 1-15.

[8] **Dvurečenskij A and Pulmanová S** (1998), *Quotiens of partial abelian monoids and the Riesz descomposition property.* Algebra Univers. 47:443-447.

[9] **Dvurečenskij A and Pulmanová S**, *New Trends in quantum Structures*, Kluwer Academic Publishers, Dordrecht (2000).

[10] **Dvurečenskij A Vetterlein Th** (2001), *Pseudoeffect algebras I. Basic Properties*, International Journal of Theoretical Physics, Vol. 40, No. 3.

[11] **Dvurečenskij A Vetterlein Th** (2001), *Pseudoeffect algebras I. Group Representations*, International Journal of Theoretical Physics, Vol. 40, No. 3.

[12] **Dwinger, Ph**, *Introduction to Boolean Algebras*, Physica-Verlag, Würzburg (1967).

[13] **Foulis D J**, *Logic and Partially Abelian Groups*, Preprint, ArXiv.org, math. LO/0504553 vl, 27 Apr 2005.

[14] **Foulis D J and Bennett M K** (1994), *Effect Algebras and Unsharp Quantum Logics*, Foundations of Physics, 24:1331-1352.

[15] **Gudder S and Pulmannová S** (1997), *Quotiens of partial abelian monoids*. Algebra Univers. 38:395-421.

[16] **Halmos P**, *Lectures on Boolean Algebras*, Van Nostrand, New York (1972).

[17] **Jenča G** (2007),*A representation theorem for MV-Algebras*. Soft. Comput. 11:557-564.

[18] **Monk J D**(Editor),*Handbook of Boolean Algebras*, 3 vol., North-Holland, Amsterdam (1989).

[19] **Rasiowa H and Sikorski R**, *The mathematics of Metamathematics*, Polish Scietific Publishers, Warsaw (1963).

[20] **Sikorski R**, *Boolean Algebras*, third edition, Springer-Verlag, New York (1969).

[21] **Sikorski R**, **Álgebras de Boole**, Notas de Lógica Matemática No. 4, Universidad Nacional del Sur, Bahía Blanca (1968).