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Teoría de Singularidades

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Introducción

Durante las décadas de los 60 y 70, la teoría de las catástrofes de Thom, inspirada en el previo trabajo de Whitney sobre puntos críticos, surgió como un área de la matemática destinada al estudio de sistemas dinámicos. Buscaba, entre otras cosas, entender grandes cambios en el comportamiento de ciertos procesos, ocasionados por pequeños cambios en las circunstancias que lo rodean, analizando la dependencia de soluciones de ecuaciones respecto de sus parámetros, a través del estudio de puntos críticos degenerados. En esta dirección, Vladimir Arnold comenzó a utilizar el término *teoría de singularidades*, para referirse al área que mezclaba ideas de los trabajos de Whitney, Thom y otros sobre teoría de las catástrofes, con aportes de la geometría algebraica. Así, la teoría de singularidades es una herramienta para el estudio de fenómenos abruptos que ocurren en sistemas que dependen de parámetros de manera suave. Sus aplicaciones van desde la teoría de sistemas dinámicos, hasta otras áreas como óptica, mecánica cuántica, geometría algebraica y topología diferencial, por mencionar algunas. En su trabajo, define una relación de equivalencia entre gérmenes de funciones (diferenciables u holomorfas) que identifica dos gérmenes, si existen cambios de coordenadas que lleven una función a la otra. Así, una singularidad es la clase de equivalencia (vía esta relación) de un germen de punto crítico. Uno de los grandes éxitos de la teoría desarrollada por Vladimir Arnold (y explicada en profundidad en [1]) fue la clasificación de singularidades de puntos críticos de funciones vía la reducción a sus formas normales.

En este trabajo, estudiamos la teoría de singularidades desarrollada por Vladimir Arnold (con énfasis en el caso holomorfo, pero siempre teniendo en mente que los resultados se aplican también al caso diferenciable con métodos similares pero con modificaciones técnicas no triviales), llegando a mostrar algunas de las técnicas usadas en la clasificación de singularidades y haciendo el segmento inicial de la clasificación (es decir, dando formas normales para las singularidades *simples*). Las principales referencias para este trabajo son los textos [1] y [3].

En el capítulo 1 definimos el espacio de jets, que servirá como marco para el estudio de las singularidades y probamos algunos resultados que servirán como herramientas en próximos capítulos. También, enunciamos el problema de equivalencia entre funciones y el concepto de singularidad, mencionando algunos de sus invariantes. Probamos el Splitting Lemma, que servirá para comparar singularidades de funciones con distinto número de variables.

En el capítulo 2, estudiamos el álgebra local y la multiplicidad (también llamada número de Milnor) como invariantes bajo la relación de equivalencia. Probaremos que una singularidad es aislada si y sólo si la dimensión del álgebra local es finita y el teorema de determinación finita de Tougeron (que dice que una singularidad aislada es equivalente a un polinomio).

En el capítulo 3, estudiamos las deformaciones versales de singularidades. Probaremos el teorema de versalidad, que da condiciones infinitesimales para que una deformación sea versal. Introduciremos los conceptos de modalidad y forma normal.

En el capítulo 4, estudiamos las funciones quasihomogéneas y semi quasihomogéneas, que servirán como herramienta para dar formas normales de singularidades de modalidad baja. Arnold dio la clasificación completa para modalidad menor o igual a 2, ver [3]; en este trabajo mostraremos completamente el caso de modalidad 0. Definimos el diagrama de

Newton de una serie de potencias y damos un teorema sobre formas normales de funciones semi quasihomogéneas.

En el capítulo 5, utilizamos lo desarrollado en capítulos anteriores para dar las formas normales de las singularidades simples, mediante métodos que sirven también en la clasificación de singularidades de modalidad superior. Esto en particular, demuestra la clasificación de catástrofes elementales de Thom, además de darle una interpretación ADE.

Introduction

During the 1960s and 1970s, Thom's catastrophe theory, inspired in the previous work of Whitney on critical points, emerged as an area for the study of dynamical systems. Its goal was to understand big shifts in the behaviour of certain processes caused by small changes in circumstances, by analizing the dependence of solutions of equations on the parameters appearing in them and studying degenerate critical points. In this direction, Vladimir Arnold started to use the term *singularity theory* to refer to the area that mixed the ideas of the work of Whitney, Thom and catastrophe theory, with some input from algebraic geometry. Therefore, singularity theory is a tool for the study of abrupt, jump-like phenomena, occurring in systems that depend smoothly on parameters. It has applications in many areas such as the theory of dynamical systems, optics, quantum mechanincs, algebraic geometry and differential topology. In its work, Arnold defines an equivalence relation between germs of functions (smooth or holomorphic) that identifies two germs if there exist coordinate changes bringing one function to the other. A singularity is an equivalence class of a germ of critical point. One of the big results of the theory developed by Arnold (and explained in [1]) was the classification of singularities of critical points of functions via reduction to its normal forms.

In this work, we study singularity theory, as developed by Vladimir Arnold (making emphasis in the holomorphic case, but always having in mind that most results apply also for smooth funcitons, with technical and non-trivial modifications) and show some of the techniques used in the classification of singularities. We will make the initial segment of this classification, by showing normal forms for simple singularities. The main references for this work are [1] and [3].

In chapter 1, we define the jet spaces, that will be used as a framework for the study of singularities and we prove results that will be useful tools for next chapters. Also, we introduce the problem of equivalence between functions and the concept of singularity, mentioning some of its invariants. We prove the Splitting lemma, that will be useful to compare singularities of functions with different number of variables in the source space.

In chapter 2, we study the local algebra and multiplicity (also called *Milnor number*) as invariants under the equivalence relation. We prove that a singularity is isolated if and only if the dimension of the local algebra is finite and Tougeron's finite determinacy theorem (which states that an isolated singularity is equivalent to a polynomial).

In chapter 3, we study versal deformations of singularities. We prove a theorem that gives infintesimal conditions for a deformation to be versal, called the versality theorem. We introduce the concepts of modality and normal form.

In chapter 4, we study quasihomogeneous and semi quasihomogeneous functions, that will be useful tools to show normal forms of singularities of low modality. Arnold gave the complete classification for modality less or equal than 2, see [3]. We define the Newton diagram of a power series and give a theorem on normal forms for semi quasihomogeneous functions.

In chapter 5, we use the results of former chapters to give normal forms of simple singularities, using methods that are useful also in the classification of singularities of higher modality. This in particular proves Thom's theorem of clasification of elementary catastrophes and gives it an ADE interpretation.

Contents

Chapter 1

Preliminaries

In this chapter, we introduce the basic concepts of germs of functions and jet spaces that will allow us to work with the classification of singularities. We also define singularities and state the problem of its classification. As an initial way to attack this problem, we will prove some theorems such as the Morse lemma and the Splitting lemma. We will also state and prove some Preparation theorems, which will be a powerful tool for our study of deformations and isolatedness of singularities. These results and the proofs have been taken from different sources, such as [8], [7] and [4].

1.1 Germs of functions and Jet spaces

Definition 1.1.1. Let $n, m \in \mathbb{N}$ and $p \in \mathbb{C}^n$ and define a relation in the space of holomorphic maps $\{f : U \to \mathbb{C}^m : f \text{ is holomorphic}, U \subseteq \mathbb{C}^n \text{ is open and } p \in U\}$, such that $f : U \to \mathbb{C}^m \simeq g : V \to \mathbb{C}^m$ if $f|_{U\cap V} = g|_{U\cap V}$ (here, $U, V \subseteq \mathbb{C}^n$ are open sets that contain p). This is an equivalence relation in the space of holomorphic maps defined in a neighborhood of p. An equivalence class of this relation is called a *map-germ*. The equivalence class of a map f at p will be written as $f : (\mathbb{C}^n, p) \to (\mathbb{C}^m, f(p))$ or just $\overline{f}_p$ (if there is no confusion, we may also call f to its equivalence class). The set of all map-germs at p will be denoted $\mathscr{O}_{n,m;p}$. If $m = 1$, we will denote $\mathscr{O}_{n;p}$ and if $p = 0$, we will just write $\mathscr{O}_n$.

Remark 1.1.2.

- The same definition can be made for maps between complex or differentiable manifolds, since the definition is local.
- The space of holomorphic maps $\mathscr{O}_{n;p}$ is an algebra. Indeed, we can define the sum and product of maps in the intersection of their domains: $\overline{f}_p + \overline{g}_p = \overline{f+g}_p$, $\overline{f}_p\overline{g}_p = \overline{fg}_p$ and $\lambda\overline{f}_p = \overline{\lambda f}_p$ for $\lambda \in \mathbb{C}$ and f, g holomorphic maps defined in a neighborhood of p. It is easy to check that it is an algebra, since the space of holomorphic functions is. Also, it is a local algebra: the ideal $\mathfrak{m}_n = \{f : (\mathbb{C}^n, p) \to (\mathbb{C}, 0)\}$ of maps that vanish in p is the only maximal ideal: if $U \subseteq \mathbb{C}^n$ is open, $p \in U$, $g : U \to \mathbb{C}$ is holomorphic and $g(p) \neq 0$, then there is an open neighborhood $V \subseteq U$ of p such that $g|_V$ does not vanish. Therefore, $\overline{g}_p\overline{\left(\frac{1}{g|_V}\right)}_p = \overline{1}_p$ which means that $\overline{g}$ is a unit, and thus $\mathfrak{m}_n^c = \{\text{Units of } \mathscr{O}_{n;p}\}$.

Definition 1.1.3. We call the space $J^k_p(\mathbb{C}^n) = \mathscr{O}_{n;p}/\mathfrak{m}^k_{n;p}$ the space of k-jets of map-germs defined on p. We denote the natural projection by $j^k_p : \mathscr{O}_{n;p} \to J^k_p(\mathbb{C}^n)$ (and we do not write the p when $p = 0$).

We will see that the space of k-jets codifies the taylor polynomials up to order k and has a differentiable structure.

Lemma 1.1.4 (Hadamard)**.** *Let $f : \mathbb{C}^{n+k} \to \mathbb{C}$ be a holomorphic function and $x \in \mathbb{C}^n, y \in \mathbb{C}^k$ be such that (x, y) are the coordinates of the domain of f. Then there exist holomorphic functions $g_1, \dots, g_n : \mathbb{C}^{n+k} \to \mathbb{C}$ such that $g_i(x, 0) = \frac{\partial f}{\partial y_i}(x, 0)$ and*

$$f(x, y) - f(x, 0) = \sum_{i=1}^{k} y_i g_i(x, y).$$

Proof. Just note that

$$f(x, y) - f(x, 0) = \int_0^1 \frac{\partial}{\partial t} f(x, ty)dt = \sum_{i=1}^{k} y_i \int_0^1 \frac{\partial f}{\partial y_i}(x, ty)dt.$$

□

If $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $f : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function, we will denote $\frac{\partial^{|\alpha|} f}{\partial z^\alpha} = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$.

Corolary 1.1.5. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, then for every $k \in \mathbb{N}$, there exist functions $c_\alpha : \mathbb{C}^n \to \mathbb{C}$ such that*

$$f(z) = T_{k-1}(z) + \sum_{|\alpha|=k} c_\alpha(z) z^\alpha$$

where $T_{k-1}(z)$ is the $k-1$ Taylor polynomial of f and $c_\alpha(0) = \frac{1}{k!}\frac{\partial^k f}{\partial x^\alpha}(0)$.

Proof. The case $k = 1$ is exactly Hadamard's Lemma 1.1.4. For $k > 1$ we use induction. Indeed, our hypothesis says that

$$f(z) = T_{k-2}(z) + \sum_{|\beta|=k-1} c_\beta(z) z^\beta \tag{1.1}$$

with $c_\beta(0) = \frac{1}{(k-1)!}\frac{\partial^{k-1} f}{\partial x^\beta}(0)$. Using Hadamard's Lemma 1.1.4 over each c_β, we get

$$c_\beta(z) = \frac{1}{(k-1)!}\frac{\partial^{k-1} f}{\partial x^\beta}(0) + \sum_{i=1}^{n} g_{\beta,i}(z) z_i$$

where $g_{\beta,i}(0) = \frac{\partial c_\beta}{\partial z_i}(0)$. Replacing this in 1.1 we get the desired formula for k, completing the induction (to see that $g_{\beta,i}(0) = \frac{1}{k!}\frac{\partial^k f}{\partial x^\beta \partial x_i}(0)$, we just take $\frac{\partial^k}{\partial x^\beta \partial x_i}$ both sides of the equation and evaluate in 0). □

Lemma 1.1.6. *The map* $J^k_p(\mathbb{C}^n) \xrightarrow{\gamma} \mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^k$ *such that*

$$\gamma(f) = \sum_{s=0}^{k} \sum_{1 \leq i_1, \ldots, i_k \leq k} \frac{1}{k!} \frac{\partial^k f}{\partial x_{i_1} \cdots x_{i_k}}(p)(x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k})$$

is an isomorphism.

Proof. Injectivity is immediate from our previous computation and it is obvioulsy surjective (since $\mathbb{C}[x_1, \ldots, x_n] \subseteq \mathscr{O}_n$). □

Remark 1.1.7. We can give the space $J^k_p(\mathbb{C}^n)$ a structure of complex manifold. After our last identification, it is clear that the space of k-jets is a $\mathbb{C}$ vector space of finite dimension, and a basis is given by the monomials of degree less than k.

1.2 Definitions, Morse lemma and Splitting lemma

Definition 1.2.1. Let $U \subseteq \mathbb{C}^n$ an open set and $f : U \to \mathbb{C}$ an holomorphic function. A critical point is said to be *nondegenerate* or *Morse critical point* if the second differential is a nondegenerate quadratic form (or equivalently, its Hessian matrix is invertible). The *corank* of a critical point is defined as the dimension of the kernel of the second differential. Morse critical points have corank 0.

Definition 1.2.2. Let $f, g : (\mathbb{C}^n, 0) \to \mathbb{C}$ be two holomorphic function-germs. We say that they are *holomorphically equivalent* (or just equivalent if there is no confusion) if there exists a biholomorphism $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = g \circ h$, making the following a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{f} & \mathbb{C} \\ \downarrow h & \nearrow g & \\ (\mathbb{C}^n, 0) & & \end{array} .$$

Clearly, this is an equivalence relation. By precomposing our function with a translation, we can always assume that the function has a critical point at 0. The equivalence class of a function-germ at a critical point is called a *singularity*.

Remark 1.2.3.
- The same definition can be given for maps instead of function-germs, and for maps between differentiable manifolds M and N, although generally the equivalence between to maps $f, g : M \to N$ is given by the existence of diffeomorphisms in both the source and target space that make the following a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \simeq & & \downarrow \simeq \\ M & \xrightarrow{g} & N \end{array} .$$

- Let $\mathscr{D}_n$ the group of biholomorphic map-germs $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. This group acts in the space $\mathscr{O}_n$ of holomorphic function-germs in 0: if $g \in \mathscr{D}_n$ and $f \in \mathscr{O}_n$, then we define the action by $g \cdot f = f \circ g^{-1}$. The orbits of this action are exactly the equivalence classes defined before. Thus, the classification of singularities consists in classifying the orbits of this action.

- The corank is an invariant of a singularity: equivalent function-germs have equal coranks. If $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ has a critical point at 0 and $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a biholomorphism, then

$$\frac{\partial^2 (f \circ h)}{\partial x_j \partial x_i} = \frac{\partial^2 (f \circ h)}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \frac{\partial h_k}{\partial x_i} \right) = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial h_k}{\partial x_i} + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \frac{\partial^2 h_k}{\partial x_j \partial x_i}$$

  where $h_k : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is the k-th component of h. Since f has a critical point at 0, we conclude that $H(f \circ h)(0) = H(f)(0) \cdot Dh(0)$, where $H(f)$ is the hessian matrix of f. The fact that $Dh(0)$ is invertible says that f and $f \circ h$ have the same corank.

The classification of the singularities of non-degenerate critical points is given by the Morse lemma.

Lemma 1.2.4 (Morse)**.** *In a neighborhood U of a Morse critical point $p \in \mathbb{C}^n$ of a function $f : \mathbb{C}^n \to \mathbb{C}$, there is a biholomorphism $g : U \to V$ such that*

$$f(g(x_1, \ldots, x_n)) = f(p) + x_1^2 + \cdots x_n^2.$$

We will prove two generalizations of this lemma: see 1.2.5 and 2.3.3.

From the Invariance of Domain theorem, it is clear that two equivalent function-germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}, g : (\mathbb{C}^m, 0) \to \mathbb{C}$ must satisfy $n = m$. However, some functions of different number of variables "behave" similarly around a critical point. This is exactly the content of the Splitting lemma (also called parametric Morse lemma, as it generalizes the Morse lemma).

Theorem 1.2.5 (Splitting lemma)**.** *In a neighborhood of the critical point* 0 *of corank k, a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivalent to a function of the form*

$$\varphi(x_1, \ldots, x_k) + x_{k+1}^2 + \cdots + x_n^2$$

where $\mathfrak{m}$ is the (only) maximal ideal of maps vanishing in the origin and $\varphi \in \mathfrak{m}^3 \subseteq \mathscr{O}_n$.

Proof. Since the Hessian matrix of f is symmetric, we can make a lineal change of coordinates $u = u(x)$ such that the hessian matrix of f has the form

$$\left(\begin{array}{cccc|ccc}
1 & 0 & \ldots & 0 & & 0 & \\
0 & \ddots & \ddots & \vdots & & \vdots & \\
\vdots & \ddots & \ddots & 0 & & \vdots & \\
0 & \ldots & 0 & 1 & & 0 & \\
\hline
 & & & & 0 & & \\
0 & \ldots & \ldots & 0 & & \ddots & \\
 & & & & & & 0
\end{array}\right)$$

where the top left block of the matrix has size $(n-k)\times(n-k)$.

Thus, using the Implicit Function Theorem, the set $\left\{\frac{\partial f}{\partial u_1}=\cdots=\frac{\partial f}{\partial u_{n-k}}\right\}$ can be expressed locally as the graph of a holomorphic function $g:\mathbb{C}^k\to\mathbb{C}^{n-k}$ (putting the image of g in the first $n-k$ coordinates). Now, let us call $\phi:\mathbb{C}^n\to\mathbb{C}^n$ such that $\phi(u_1,\ldots,u_n)=(u_1,\ldots,u_n)+(g(u_{n-k+1},\ldots,u_n),0)$. It is clearly a local biholomorphism since its jacobian at 0 has the form

$$\left(\begin{array}{c|c} Id & * \\ \hline 0 & Id \end{array}\right).$$

Now, let $F=f\circ\phi$. If we fix the last k coordinates, the function $F_{(u_{n-k+1},\ldots,u_n)}:\mathbb{C}^{n-k}\to\mathbb{C}^n$ such that $F_{(u_{n-k+1},\ldots,u_n)}(u_1,\ldots,u_{n-k})=F(u_1,\ldots,u_{n-k},u_{n-k+1},\ldots,u_n)$ has a nondegenerate critical point at the origin. We write $\varphi(u_{n-k+1},\ldots,u_n)=F(0,\ldots,0,u_{n-k+1},\ldots,u_n)$. Now, using Hadamard's Lemma 1.1.4 (and having in mind that the dependence on the remaining parametets is still holomorphic) we get that

$$F(u_1,\ldots,u_{n-k},u_{n-k+1},\ldots,u_n)-\varphi(u_{n-k+1},\ldots,u_n)=\sum_{i=1}^{n-k}u_i g_i^{(u_{n-k+1},\ldots,u_n)}(u_1,\ldots,u_{n-k})$$

where the dependance of the last variables of the g_i is holomorphic. Since

$$g_i^{(u_{n-k+1},\ldots,u_n)}(0)=\frac{\partial F}{\partial x_i}(0,\ldots,0,u_{n-k+1},\ldots,u_n))=0$$

holds, we use again Hadamard's lemma 1.1.4 over each g_i to get

$$g_i^{(u_{n-k+1},\ldots,u_n)}(u_1,\ldots,u_{n-k})=\sum_{j=1}^{n-k}u_j h_{i,j}^{(u_{n-k+1},\ldots,u_n)}(u_1,\ldots,u_{n-k})$$

(and again, the dependance of the last k variables is holomorphic). Hence,

$$F(u_1,\ldots,u_{n-k},u_{n-k+1},\ldots,u_n)-\varphi(u_{n-k+1},\ldots,u_n)=\sum_{i,j=1}^{n-k}u_iu_j h_{i,j}^{(u_{n-k+1},\ldots,u_n)}(u_1,\ldots,u_{n-k}). \tag{1.2}$$

Our goal now will be to prove that the right-hand side is equivalent to a sum of quadratic forms for each $(u_{n-k+1},\ldots,u_n)$, depending holomorphically on this variables.

In the last equation (due to symmetry of indices i,j) we replace $h_{i,j}^{(u_{n-k+1},\ldots,u_n)}$ by the average

$$\frac{h_{i,j}^{(u_{n-k+1},\ldots,u_n)}+h_{j,i}^{(u_{n-k+1},\ldots,u_n)}}{2}$$

so that the matrix $h_{i,j}$ be symmetric. Also, differentiating twice in 1.2, we get that $2h_{i,j}^{(u_{n-k+1},\ldots,u_n)}(0)=\frac{\partial^2 F}{\partial x_i\partial x_j}(0,\ldots,0,u_{n-k+1},\ldots,u_n)$, that is an invertible matrix (moving the indices i,j).

Now, making a linear change of coordinates, we can assume that $h_{1,1}^{(u_{n-k+1},\ldots,u_n)}(0) \neq 0$, and let g be a square root of $h_{1,1}^{(u_{n-k+1},\ldots,u_n)}$ around 0. We make a change of coordinates

$$\begin{aligned} v_1 &= g(u_1,\ldots,u_{n-k})\left(u_1 + \sum_{i=2}^{n-k} u_i \frac{h_{i,1}}{h_{1,1}}(u_1,\ldots,u_{n-k})\right) \\ v_i &= u_i, i \geq 2. \end{aligned}$$

By the inverse function theorem, it is a local biholomorphism. Indeed, the matrix of the change is triangular with 1s on the diagonal except for $\frac{\partial v_1}{\partial u_1}(0) = g(0) \neq 0$. Now, we get that

$$\sum_{i,j=1}^{n-k} v_i v_j h_{i,j}^{(u_{n-k+1},\ldots,u_n)}(v_1,\ldots,v_{n-k}) = v_1^2 + \sum_{i,j=2}^{n-k} v_i v_j (h')_{i,j}^{(u_{n-k+1},\ldots,u_n)}(v_1,\ldots,v_{n-k})$$

where the $(h')_{i,j}$ are other functions that depend holomorphically of all the variables (even the last ones). Repeating this procedure $n-k$ times, we get the desired decomposition. □

This lemma says that the behaviour of a function near a critical point of corank k can be found by studying a function of k variables, independently of the number of variables of the function. The reduction of variables is what makes the Splitting lemma so useful.

This motivates a very natural definition of a measure of degeneracies of critical points of functions of different numbers of variable.

Definition 1.2.6. Two function-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ are said to be *stably equivalent* if they become equivalent after the addition of nondegenerate forms in supplementary variables:

$$f(x_1,\ldots,x_n) + x_{n+1}^2 + \cdots + x_k^2 \simeq g(y_1,\ldots,y_m) + y_{m+1}^2 + \cdots + y_k^2.$$

And this new notion of equivalence is coherent with our former one.

Theorem 1.2.7. *Two functions-germs of the same number of variables are stably equivalent if and only if they are equivalent*

Proof. This result can be found in [1] Chapter 1, Section 1.3

□

So, from now on, we will classify the singularities of function germs up to stable equivalence, which allows us to identify critical points of functions of different variables that behave similarly in a neighborhood of the critical point.

Example 1.2.8. The funcion germs of $f(z_1) = z_1^3$ and $g(w_1, w_2, w_3) = w_1^3 + w_2 w_3$ at 0 are stably equivalent critical points, since $h(w_1, w_2, w_3) = (z_1, z_2 + iz_3, z_2 - iz_3)$ gives the equivalence between $f(z_1) + z_2^2 + z_3^2$ and g.

1.3 Preparation theorems

Proposition 1.3.1. *Let R be a commutative local ring with unit and its only maximal ideal $\mathfrak{m}$. If $B \in M_n(\mathfrak{m})$, and* Id *is the $n \times n$ identity matrix, then $Id + B$ is an invertible matrix.*

Proof. We use induction in n. If $n = 1$, $1 + b$ is a unit (with $b \in \mathfrak{m}$) because it does not belong to $\mathfrak{m}$. For $n > 1$, let us compute the determinant of $Id+B$ by expanding by the first row. Let $M_{i,j}$ be the determinant of the $(n-1) \times (n-1)$ minor of $Id+B$ that results from the elimination of the i-th row and j-th column of $Id + B$ (defined for every $1 \leq i, j \leq n$). Therefore, we have

$$|Id + B| = (1 + B_{11})M_{1,1} + \sum_{i=2}^{n}(-1)^{i+1}B_{1,i}M_{1,i}.$$

Since each $B_{ij} \in \mathfrak{m}$, then $|Id + B|$ is a unit if and only if $M_{1,1}$ is. Since it is a $(n-1) \times (n-1)$ matrix that is also of the form $Id + C$ where C has coefficients in $\mathfrak{m}$, the induction says that $|Id + B|$ is a unit and thus $Id + B$ is invertible. □

Lemma 1.3.2 (Nakayama)**.** *If M is a finitely generated module over R a local and commutative ring with unit, such that $M = \mathfrak{m}M$, then $M = 0$.*

Proof. Let $a_1, \ldots, a_n$ be generators of M. We know that there exist a matrix $B = (B_{ij}) \in M_n(\mathfrak{m})$ such that $a_i = \sum B_{ij}a_j$ for every $1 \leq i \leq n$. Then, we have a system

$$A = BA \Leftrightarrow (Id - B)A = 0.$$

Using 1.3.1, we know that $Id - B$ is invertible and thus each $a_i = 0$. □

Theorem 1.3.3 (Weierstrass Preparation Theorem)**.** *Let $f : (\mathbb{C}^{m+1}, 0) \to \mathbb{C}$ be a holomorphic function-germ at 0, and let $z \in \mathbb{C}^m$ and $w \in \mathbb{C}$ be coordinates such that $f = f(z, w)$. If $f(0, w)$ is a monic polynomial on w of degree n, then there exists a holomorphic function-germ $h : (\mathbb{C}^{m+1}, 0) \to \mathbb{C}$ such that $h \neq 0$ and holomorphic function-germs $a_1, \ldots, a_n : (\mathbb{C}, 0) \to \mathbb{C}$ such that*

$$f = gh;\ g(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z).$$

We call $g(z, w)$ the Weierstrass polynomial.

Proof. Denote $b_i(z)$ the zeros of the function $f(z, \bullet) : B_\varepsilon(0) \subseteq \mathbb{C} \to \mathbb{C}$ for every z in a neighborhood of $z = 0$ (probably repeated, according to their multiplicity), such that $f(z, \bullet)$ is defined in $B_\varepsilon(0)$. In fact, there exists a neighborhood V of $z = 0$ such that the number of zeros of the function $f(z, \bullet)$ is constant in V. This is because $f(z, w) \xrightarrow{z \to 0} f(0, w)$ and

$$\int_{|u|=\varepsilon} \frac{\frac{\partial f}{\partial w}(z, u)}{f(z, u)} du$$

counts the number of zeros of $f(z, \bullet)$ in $B_\varepsilon(0)$. Therefore, we can write $f(z,w) = h(z,w)\prod(w-b_i(z))$, where $h \neq 0$. Our candidate to be g is $\prod(w-b_i(z))$, that has as coefficients the symmetric polynomials of the zeros $b_i(z)$. Since the ring of symmetric polynomials is generated by sums of powers of the b_i and we know (using Cauchy's formula) that

$$b_1(z)^s + \cdots + b_n(z)^s = \frac{1}{2\pi i}\int_{|z|=R} w^s \frac{\frac{\partial f}{\partial w}}{f}(z,w)dw.$$

This says that the functions $a_i(z)$ and g are holomorphic in a neighborhood of the origin. Finally, h is holomorphic for small $|z|, |w|$ since

$$h(z,w) = \frac{1}{2\pi i}\int_{|u|=R} \frac{h(z,u)}{u-w}du = \frac{1}{2\pi i}\int_{|u|=R} \frac{\frac{f(z,u)}{g(z,u)}}{u-w}du.$$

□

Theorem 1.3.4 (Division Theorem). *Let $f(z,w)$ be as in 1.3.3. Then for any germ of holomorphic function $\phi(z,w)$, there exist holomorphic germs $h(z,w)$ and $h_i(z), 1 \leq i \leq n-1$ such that*

$$\phi = hf + \sum_{i=0}^{n-1} h_i(z)w^i.$$

Proof. Using the 1.3.3, we can assume that f is a Weierstrass polynomial. If we define h to be

$$h(z,w) = \frac{1}{2\pi i}\int_{|u|=R} \frac{\phi(z,u)}{f(z,u)}\frac{1}{u-w}du$$

we get that

$$\phi(z,w) - h(z,w)f(z,w) = \frac{1}{2\pi i}\int_{|u|=R} \frac{\phi(z,u)}{f(z,u)}\frac{f(z,u)-f(z,w)}{u-w}du.$$

If f is a Weierstrass polynomial of degree n in w, then $\frac{f(z,u)-f(z,w)}{u-w}$ is a polynomial of degree $n-1$ in w with holomorphic coefficients in z and u. Thus, the linearity of the integral implies the desired decomposition for ϕ. □

Now, we will prove a preparation theorem, that will be an important technical tool in our study of singularities. It is a theorem that allow to extend a solution of a functional equation along the parameters of a deformation (this will be better understood in Chapter 3, with the proof of 3.0.11).

Theorem 1.3.5 (Thom-Martinet Preparation Theorem). *Let $(x,y) \in \mathbb{C}^n \times \mathbb{C}^k$ and let $I \subseteq \mathscr{O}_{n+k}$ be an ideal. Denote $I_{x,0} = \{f(x,0) : f \in I\}$. If $e_1, \ldots, e_r \in \mathscr{O}_{n+k}$ are such that $e_1(x,0), \ldots, e_r(x,0)$ generate $\mathscr{O}_n/I_{x,0}$ as a $\mathbb{C}$ vector space, then the functions $e_1, \ldots, e_r$ generate the module $\mathscr{O}_{n+k}/I$ over $\mathscr{O}_k$. That is, for every $h \in \mathscr{O}_{n+k}$, there exist germs $g_1(y), \ldots, g_r(y)$ such that*

$$h(x,y) = \sum_{i=1}^{r} g_i(y)e_i(x,y)(mod\ I).$$

Proof. We know from the hypothesis that for every $h \in \mathscr{O}_{n+k}$, we can write

$$h(x,0) = \sum_{i=1}^{r} a_i e_i(x,o) \,(\mathrm{mod}\ I_{x,0})\,, a_i \in \mathbb{C}.$$

Using Hadamard's lemma 1.1.4 to $h(x,y)$ around $y=0$, we get that

$$h(x,y) = \sum_{i=1}^{r} a_i e_i(x,0) + f(x,0) + \mathfrak{m}_k \mathscr{O}_{n+k} h(x,y) = \sum_{i=1}^{r} a_i e_i(x,y) + f(x,y) + \mathfrak{m}_k \mathscr{O}_{n+k} \quad (1.3)$$

where $f \in I$, and we used that $f(x,y) - f(x,0), e(x,y) - e(x,0) \in \mathfrak{m}_k \mathscr{O}_{n+k}$. If we call $M = \mathscr{O}_{n+k}/I$, we know that it is a finitely generated module over $\mathscr{O}_{n+k}$, but our goal is to prove that it is finitely generated over $\mathscr{O}_k$ by the $e_1, \ldots, e_r$. If we name N the $\mathscr{O}_k$-submodule of M generated by the $e_1, \ldots, e_r$, the last equation tells us that $M = N + \mathfrak{m}_k M$ and thus $M/N = \mathfrak{m}_k M/N$. If we know M is finitely generated over $\mathscr{O}_k$, using Nakayama's Lemma 1.3.2, we get that $M/N = 0$ or $M = N$. Then, from now on we will try to prove that M is a finitely generated $\mathscr{O}_k$-module.

We make induction in n. Let $n = 1$. Because of 1.3, we know that M is finitely generated over $R = \mathscr{O}_k + \mathfrak{m}_k \mathscr{O}_{k+1}$, and its generators are $e_1, \ldots, e_r$. Now, consider the operation of multiplying by x in the R-module M. Then, there exist coefficients $B_{i,j} \in R$ such that $x e_i = \sum_{j=1}^{r} B_{i,j} e_j$, or equivalently, a matrix B with coefficients $B_{i,j}$ such that

$$(x\,\mathrm{Id} - B)E = 0, E = (e_1, \ldots, e_r).$$

Cramer's rule applied to this system says that $\det(x\,\mathrm{Id} - B) e_i = 0$ for every $1 \le i \le r$. That means $\det(x\,\mathrm{Id} - B) := \alpha(x,y)$ is an annihilator of M (and is also a monic polynomial on x of degree r with coefficients in R). It satisfies the hypothesis of the Division Theorem 1.3.4.

Finally, let $m \in M$ and let $m_1, \ldots, m_l$ be generators of M over $\mathscr{O}_{1+k}$. Then, $m = \sum_{i=1}^{l} c_i m_i, c_i \in \mathscr{O}_{1+k}$. Applying the Division Theorem to the the c_i, and then multiplying by m_i and summing over i, we obtain

$$c_i = h_i \alpha + \sum_{j=0}^{n-1} d_{i,j} x^j, d_{i,j} \in \mathscr{O}_k, h_i \in \mathscr{O}_{1+k}.$$

Also,

$$m = \sum_{i=1}^{l} \sum_{j=0}^{n-1} d_{i,j} m_i x^j$$

because $\alpha m_i = 0$. This says that $x^j m_i, 1 \le i \le l, 0 \le j \le n-1$ generate M over $\mathscr{O}_k$.

To finish the induction, assume that $n > 1$ and $M = N + \mathfrak{m}_k M$. This implies that $M = \mathscr{O}_{n-1+k} + \mathfrak{m}_{n-1+k} M$. Applying the first induction step, we get that M is finitely generated as $\mathscr{O}_{n-1+k}$-module. The inductive hypothesis implies that it is finitely generated over $\mathscr{O}_k$, and thus completes the proof. □

Chapter 2

Local algebra of a map

Every geometric object can be described in two ways: in terms of points of manifolds and in terms of the functions on them. The algebraic way of describing the geometric objects (that is, via the algebra of functions on the manifold) becomes very useful when describing singularities, because of the difficulties arising from their infinitesimal nature.

In this chapter, we introduce an important invariant of an holomorphic function germ: the local algebra. We will prove that the dimension of that algebra (also known as the *Milnor number*), seen as a complex vector space, is equal to the index of the function in the point. In addition, it will allow us to characterize the isolated singularities and will play an important role in the proof of Tougeron's finite determinacy theorem at the end of the chapter. This surprising result says that complex isolated singularities have a polynomial representative in its class (one of its Taylor's polynomials), and is very useful in the classification of singularities.

The exposition of these topics follows [3], Part I, Chapters 5 and 6.

2.1 Definitions

Definition 2.1.1. Let $f : (\mathbb{C}^n, a) \to (\mathbb{C}^m, 0)$ be a germ of a holomorphic function, $a \in \mathbb{C}^m$. The *local algebra of the map f at a* is the quotient algebra of the function-germs by the ideal generated by the components of the map, which we call $I_{f,a} = \langle f_1, \dots, f_m \rangle$. We denote it $Q_{f,a} = \mathscr{O}_n / I_{f,a}$. Its dimension as a $\mathbb{C}$-vector space is called the *algebraic multiplicity* of f at a. If $a = 0$, we will write I_f and Q_f, for the ideal generated by the components and the local algebra respectively.

Definition 2.1.2. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with a critical point at zero. The *gradient ideal* is the ideal $I_{\nabla f} \subseteq \mathscr{O}_n$ generated by the partial derivatives of the function f. The *local algebra of the singularity* of f is $Q_{\nabla f} = \mathscr{O}_n / I_{\nabla f}$.

Remark 2.1.3. The algebra $Q_{\nabla f}$ does not depend on the choice of local coordinates. If $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a biholomorphism, then there is an exact sequence isomorphism, where vertical arrows are identities

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\nabla(f)} & \longrightarrow & \mathscr{O}_n & \longrightarrow & Q_f & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{\nabla(f\circ h)} & \longrightarrow & \mathscr{O}_n & \longrightarrow & Q_{f\circ h} & \longrightarrow & 0 \end{array}$$

since $\nabla(f\circ h)(z) = \nabla(f)(h(z))Dh(z)$ and $Dh(z)$ is invertible.

Definition 2.1.4. The *Milnor number* of the germ $f \in \mathscr{O}_n$ is the dimension of $Q_{\nabla f}$ seen as a $\mathbb{C}$-module:

$$\mu(f) = \dim_{\mathbb{C}} Q_{\nabla f}.$$

A critical point is said to be of *finite multiplicity* if $\mu(f) < \infty$ (we will see later that this is equivalent to being *isolated* 2.2.2).

Example 2.1.5.

- Let $f : \mathbb{C} \to \mathbb{C}$ such that $f(x) = x^k$. Then $Q_{f'} = \mathscr{O}_n/\langle x^{k-1}\rangle$, so f has at 0 a critical point of multiplicity $k-1$ (using Taylor's formula 1.1.5).
- Let $f : \mathbb{C}^2 \to \mathbb{C}$ such that $f(x,y) = x^2y + y^k$. The local algebra of the singularity is $\mathscr{O}_2/\langle x^2 + ky^{k-1}, 2xy\rangle$ is generated by $1, x, y, y^2, \dots, y^{k-1}$, and thus has multiplicity $k+1$.
- Let $f : \mathbb{C}^2 \to \mathbb{C}$ such that $f(x,y) = x^3 + y^4$. The local algebra of the singularity is $\mathscr{O}_2/\langle 3x^2, 4y^3\rangle$ is generated by $1, x, y, y^2, xy, xy^2$, and thus has multiplicity 6.
- Let $f : \mathbb{C}^2 \to \mathbb{C}$ such that $f(x,y) = x^3 + xy^3$. The local algebra of the singularity is $\mathscr{O}_2/\langle 3x^2 + y^3, 3xy^2\rangle$ is generated by $1, x, xy, y, y^2, y^3, y^4$ and thus has multiplicity 7.
- Let $f : \mathbb{C}^2 \to \mathbb{C}$ such that $f(x,y) = x^3 + y^5$. The local algebra of the singularity is $\mathscr{O}_2/\langle 3x^2, 5y^4\rangle$ is generated by $1, x, y, y^2, y^3, xy, xy^2, xy^3$ and thus has multiplicity 8.

Definition 2.1.6. Let a be an isolated root of a smooth map-germ $f : (\mathbb{R}^n, a) \to \mathbb{R}^n$. The *index* of f at a is

$$\mathrm{ind}_a[f] = \deg\left(\frac{f(\varepsilon x)}{||f(\varepsilon x)||} : S_1^{n-1}(a) \to S_1^{n-1}\right)$$

where S_1^{n-1} is the sphere centered in a with radius 1 and ε is sufficiently small for a to be the only root of f in $B_\varepsilon(a)$. In the holomorphic case, we can think of a map-germ $f : (\mathbb{C}^n, a) \to \mathbb{C}^n$ as a smooth map-germ $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and apply the same definition.

Remark 2.1.7. The index is well defined. Indeed, if there are no roots of f in both B_ε and $B_{\varepsilon'}$, for sufficiently small $\varepsilon, \varepsilon'$, then the maps $\frac{f(\varepsilon x)}{||f(\varepsilon x)||}$ and $\frac{f(\varepsilon' x)}{||f(\varepsilon' x)||}$ are homotopic, via the linear homotopy.

Example 2.1.8. If $f(0) = 0$ and $Df(0)$ is invertible, then $\mathrm{ind}_a[f]$ is equal to 1 or -1 depending on the sign of the jacobian. Indeed, by the inverse function theorem, there exists an inverse map $f^{-1} : \overline{B_\varepsilon(0)} \to f^{-1}(\overline{B_\varepsilon(0)})$ and $\frac{f}{||f||} \circ f^{-1} : \partial B_\varepsilon(0) \to S_1^{n-1}$ has degree 1 (it is the map that sends x to $\frac{x}{||x||}$). Thus, $\deg(\frac{f}{||f||})$ should be a unit in $\mathbb{Z}$, and it should be equal to $\deg(f^{-1}) = \deg(f)$.

Definition 2.1.9. The geometric multiplicity of a map-germ $f : \mathbb{C}^n \to \mathbb{C}^n$ at an isolated cirtical point a is the value $\text{ind}_a(f)$, that is the index of f at a.

Remark 2.1.10. Both multiplicities coincide (when well defined) for a holomorphic map germ. This will be proven later 2.2.24.

Remark 2.1.11. The same definitions can be made in the real case with practically no modification of the theory. In the holomorphic case, as we will see later, the finite algebraic multiplicity is equivalent to the isolation of the critical point. In the real case, this is not so: the function $f(x) = e^{-\frac{1}{x^2}}$ has an isolated critical point at 0 but its local algebra is infinite dimensional. Indeed, $f'(x) = \frac{2e^{-\frac{1}{x^2}}}{x^3}$, so 0 is the only critical point. And $\{x^i : i \in \mathbb{N}\}$ is a linearly independent set in $Q_{f'}$ since e is trascendent.

2.2 Local multiplicities of holomorfic maps

The main goal of this section will be to prove the next two theorems, concerning the multiplicity of map-germs at a point.

Theorem 2.2.1 (Equivalence of multiplicities)**.** *The index of a holomorphic germ of finite multiplicity is equal to its multiplicity.*

Theorem 2.2.2 (Isolatedness of roots)**.** *A holomorphic map-germ fails to be of finite multiplicity at a point a, if and only if a is a non-isolated inverse image of* 0 *of the germ.*

Definition 2.2.3. A map-germ $F : (\mathbb{C}^n, a) \to (\mathbb{C}, 0)$ is said to be *non-degenerate at a* if it has an isolated zero at a.

Remark 2.2.4. If $\delta > 0$ and $B_\delta(0) \subseteq \mathbb{R}^n$ is such that 0 is the only zero of a map-germ $f : (\mathbb{R}^n, 0) \to \mathbb{R}^n$, then the index of a point 0 is equal to the number of preimages of any sufficiently small regular value $\varepsilon \in \mathbb{R}^n$, counted with the sign of the jacobian (provided that this number of zeros is finite). Indeed, let $\{p_1, \dots, p_k\}$ be these finite zeros and let $B_1, \dots, B_n$ be mutually disjonint balls centered in each of these zeros contained in $B_\delta(0)$. Let $(f-\varepsilon)_j = \frac{f-\varepsilon}{||f-\varepsilon||} : \partial D_j \to S_1^{n-1}$. Thus, since homotopies preserve the degree, we know that

$$\text{ind}_0(f) = \sum_{j=1}^{k} \deg((f-\varepsilon)_j) + \deg\left(g = \frac{f}{||f||} : \partial X \to S_1^{n-1}\right)$$

where $X = B_\delta(0) - \bigcup_{i=1}^k \tilde{B}_i$ and $\tilde{B}_i$ is the ball B_i with the inverse orientation. By 2.1.8, we know that the $\deg((f-\varepsilon)_j)$ are 1 or -1 depending of the sign of the jacobian. And also, if w is a $n-1$ form in S_1^{n-1} that integrates 1 and $i : \partial X \to X$ is the inclusion, then

$$\begin{aligned}\deg(g) = \deg(g)\int_M w = \int_{\partial X} g^* w = \int_{\partial X} \left(\frac{f}{||f||} \circ i\right)^* w = \int_{\partial X} i^* \left(\left(\frac{f}{||f||}\right)^* w\right) \\ = \int_X d\left(\frac{f}{||f||}\right)^* w = \int_X \left(\frac{f}{||f||}\right)^* dw = 0\end{aligned}$$

since $dw = 0$ (we used Stoke's theorem).

The same argument can be used for any other deformation $H(x,t), x \in \mathbb{R}^n, t \in \mathbb{R}$ such that $H(x,0) = f$ (in this case, $H(x,t) = f(x) - t$).

Definition 2.2.5. Let $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}^n$ be two germs of smooth functions. If there exists a smooth germ $A : (\mathbb{R}^n, 0) \to GL_n(\mathbb{R})$ such that $\det(A(0)) > 0$ and $g = Af$, we say that f and g are $\mathbb{R}$-A-equivalent.

Let $f, g : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be two holomorphic function-germs. If there exists a germ of holomorphic function $A : (\mathbb{C}^n, 0) \to GL_n(\mathbb{C})$ such that $g = Af$, we say that f and g are $\mathbb{C}$-A-equivalent.

In both cases, it is an equivalence relation: If A is like in the definition and gives $f \sim g$, the function $A^{-1}(x) := A(x)^{-1}$ gives $g \sim f$.

Proposition 2.2.6. *Two $\mathbb{R}$-A-equivalent germs $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}^n$ have the same indices.*

Proof. Since $\det(A(0)) > 0$, we can join $A(0)$ and Id with a smooth path γ in $GL_n(\mathbb{R})$. As the image of that path is compact, the distance to the set $\{B \in \mathbb{R}^n : \det(B) = 0\}$ in $M_n(\mathbb{R})$ is $d > 0$. So, by the tubular neighborhood theorem we can create a tubular neighborhood $U = \{x + v : (x, v) \in N(\text{im}(\gamma)) \text{ and } |v| < \delta\}$ for that path, where $0 < \delta < d$ (thinking of $GL_n(\mathbb{R})$ embedded in $\mathbb{R}^{n^2}$ and using the notation $N(\text{im}(\gamma))$ for the normal bundle of γ). This gives a homotopy H between A and the constant map $c_{\text{Id}} : (\mathbb{R}^n, 0) \to GL_n(\mathbb{R})$ (that maps every point in $\mathbb{R}^n$ to the identity matrix). Indeed, by making a sufficiently small extension of the curve and taking a sufficiently small open in the domain of A, we can always assume that the image of A is an open ball B that belongs to the tubular neighborhood. Now, consider the vector field $X_{\gamma(t)+v} = \gamma'(t)$ defined in U (constant over each normal space) and its flow $\theta(y, s)$. We define an homotopy $G : \text{Dom}(A) \times [0, 1] \to GL_n(\mathbb{R})$ by $G(z, s) = \theta(A(z), s)$. The image of $G(z, 1)$ is an open contractible set (this is because if $(\gamma(t), v) \in N(\text{im}(\gamma))$, we know $\gamma(t) + v$ is an integral curve of X). Thus, we can build our desired H by concatenating G with an homotopy between $\text{im}(G(z, 1))$ and the identity matrix (because $G(0, 1) = \text{Id}$).

Finally, the homotopy $\overline{H}(x, t) = H(x, t) f(x)$ joins g to f and preserves the index (multiplying the function by an invertible matrix preserves the degree). ☐

Proposition 2.2.7. *If $A \in GL_n(\mathbb{C})$, then its real form $\tilde{A} \in GL_{2n}(\mathbb{R})$ has positive determinant (where $\tilde{A}(x_1, \ldots, x_{2n}) = A(x_1 + ix_2, x_3 + ix_4, \ldots, x_{2n-1} + ix_{2n})$, thinking of $\tilde{A}$ and A as linear transformations).*

Proof. It is because $\det(\tilde{A}) = |\det(A)|^2$. Since $\widetilde{AB} = \tilde{A}\tilde{B}$, we can assume that A is in Jordan form (that is because $\widetilde{CAC^{-1}} = \tilde{C}\tilde{A}\tilde{C}^{-1}$ for every invertible matrix C and so the determinant of $\tilde{A}$ and $\widetilde{CAC^{-1}}$ are equal). Thus, the determinant is the product of the eigenvalues counted with its multiplicity in the characteristic polynomial of A. If a block J_i of the Jordan form of A has the form

$$J_i = \begin{pmatrix} a_1 + ib_1 & 0 & \ldots & \ldots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 & a_n + ib_n \end{pmatrix}$$

where $a_i, b_i \in \mathbb{R}$, then we have that

$$\widetilde{J}_i = \begin{pmatrix} a_1 & -b_1 & 0 & \dots & \dots & \dots & 0 \\ b_1 & a_1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \ddots & 0 & \ddots & & \vdots \\ 0 & 1 & & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & & \ddots & 0 & 0 \\ \vdots & \ddots & & 1 & 0 & a_n & -b_n \\ 0 & \dots & 0 & 0 & 1 & b_n & a_n \end{pmatrix}.$$

Therefore, the determinant of $\widetilde{J}_i$ is $\prod_{i=1}^n (a_i^2 + b_i^2) = |\det(J_i)|^2$. If the jordan blocks of A are $J_1, \dots, J_m$, then we know that

$$\det(\tilde{A}) = \prod_{i=1}^m \det(\widetilde{J}_i) = \prod_{i=1}^m |\det(J_i)|^2 = |\det(A)|^2.$$

This completes the proof. □

Corolary 2.2.8. *$\mathbb{C}$-A-equivalent holomorphic germs have the same index.*

Proof. This is because the real forms of two holomorphic $\mathbb{C}$-A-equivalent map-germs g, f are $\mathbb{R}$-A-equivalent. This is because if $g = Af$, where $A \in GL_n(\mathbb{C})$, then $\tilde{g} = \tilde{A}\tilde{f}$, and $\tilde{A}(0)$ has positive determinant because of the previous proposition. By the real form of a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$, we mean the map $\tilde{f} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\tilde{f}(x) = (u_1(x), v_1(x) \dots u_n(x), v_n(x))$ where the components f_j of f are written as $f_j(x + iy) = u_j(x_1, y_1, \dots, x_n, y_n) + iv_j(x_1, y_1, \dots, x_n, y_n)$, where $u_j, v_j : \mathbb{R}^{2n} \to \mathbb{R}$. □

Corolary 2.2.9. *Let B be a closed ball centered in a point $a \in \mathbb{C}^n$ and f a holomorphic map defined in B such that a is the only root of f. Then, the index at a of f is equal to the number of preimages of a sufficiently small regular value ε.*

Proof. The index is equal to the number of preimages of a sufficiently small regular value $\varepsilon \neq 0$ counted with the sign of the jacobian, as discussed in 2.2.4, and we just proved that this sign is always positive. □

Proposition 2.2.10 (Additivity of the index)**.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a map with an isolated root at 0 and B a closed ball centered at 0 such that 0 is the only root of f at B. Then any sufficiently small deformation f_ε of f has finitely many zeros in B, and the sum of its indices is equal to the index of f at 0.*

Proof. If we know that the number of zeros is finite, 2.2.4 says that the sum of the indices of f_ε at these zeros is equal to the index of f at 0. So, let us see that f_ε cannot have more than $k = \operatorname{ind}_0(f)$ roots in B. If f_ε has $k+1$ different roots $a_1, \dots, a_{k+1}$ in B, let g be a polynomial that vanishes in this $k+1$ points. Then $f_\varepsilon + \delta g$ has nondegenerate roots in $a_1, \dots, a_{k+1}$ for almost every $\delta \in \mathbb{C}$. Using an analogous argument as the one in 2.2.4, we get mutually disjoint balls $B_1, \dots, B_{k+1}$ around the roots. Since these roots are nondegenerate, the degree of $f_\varepsilon + \delta g$ at each of these roots is 1 (holomorphic maps are orientation preserving). Thus,

$$k = \deg\left(\frac{f_\varepsilon}{||f_\varepsilon||}\right) = k + 1 + \deg\left(\frac{f_\varepsilon}{||f_\varepsilon||} : \partial X \to S_1^{2n-1}\right)$$

where $X = B - \bigcup_{i=1}^{k+1} \tilde{B}_i$ and $\tilde{B}_i$ are the balls B_i with the inverse orientation. Since holomorphic maps are orientation preserving and the degree counts the number of preimages of regular values with the sign of the jacobian, we get an absurd in the last equation (since the degree is non-negative and $k < k+1$). □

Remark 2.2.11. Deformations will be treated in more detail in Chapter 3. We will use this result in one particular case in the proof of our theorems, so the reader not familiar with deformations can try to use the proof of the former proposition in particular cases. (which will be convex combinations of maps).

Proposition 2.2.12. *The multiplicities of A-equivalent map germs $f, g : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ are equal.*

Proof. Indeed, since $f(x) = A(x)g(x)$, if we name $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ the coordinate functions of f and g, we have that $f_i(x) = \sum_{j=1}^n A_{ij}(x)g_j(x)$. So, $I_f \subseteq I_g$ and since it is A-equivalence is an equivalence relation, the same argument gives the other inclusion. □

Lemma 2.2.13. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a map-germ of finite multiplicity $\mu > 0$. Then, the product of any μ map-germs that vanish at 0 is contained in the ideal I_f. In particular, any monomial of degree μ or greater lies in I_f.*

Proof. Let $\varphi_1, \ldots, \varphi_\mu$ be such μ map-germs. Name $\psi_i = \prod_{j=1}^{i} \varphi_i$ for every $1 \leq i \leq \mu$ and also $\psi_0 = 1$. These $\mu + 1$ germs are linearly dependent in the ring Q_f as its dimension is μ. So, there exist elements $c_0, \ldots, c_\mu \in \mathbb{C}$ such that $\sum_{i=0}^{\mu} c_i\psi_i \in I_f$ (and not all c_i are zero). It is evident that $c_0 = 0$, otherwise I_f would contain a unit and Q_f would be trivial (which contradicts that its dimension is positive). If $r = \min\{j \in \mathbb{N} : c_j \neq 0\}$ then $\psi_r (c_r + c_{r+1}\varphi_{r+1} + \cdots + c_\mu\varphi_{r+1} \cdots \varphi_\mu) \in I_f$. This means that ψ_r belongs to I_f as $(c_r + c_{r+1}\varphi_{r+1} + \cdots + c_\mu\varphi_{r+1} \cdots \varphi_\mu)$ is invertible in $\mathscr{O}_n$. Then, $\prod_{j=1}^{\mu} \varphi_i \in I(f)$ because it is divisible by ψ_r, which completes the proof. □

Corolary 2.2.14. *A root of finite multiplicity is isolated.*

Proof. If the germ of f has finite multiplicity μ at 0, we can apply the lemma 2.2.13 to the germs x_j^μ for every $1 \leq j \leq n$. Therefore, we can write each of them in the form $\sum_{i=1}^n h_{j,i}f_i$ where $h_{j,i}$ are holomorphic map-germs. So, in a small domain (the intersection of all the domains of the $h_{j,i}$ and f_i), $f(x) = 0$ implies that $x_j^\mu = 0$ for all $1 \leq j \leq n$, and so $x = 0$. □

Corolary 2.2.15. *Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a germ of finite multiplicity μ and $g : (\mathbb{C}^n, 0) \to \mathbb{C}$ such that $f - g \in \mathfrak{m}^{\mu+1}$. Then, f and g are $\mathbb{C}$-A-equivalent.*

Proof. We name f_i and g_i to the components of f and g and we write $g_i - f_i$ as $\sum H_{i,j} f_j$ for every $1 \leq i \leq n$, where $H_{i,j}(0) = 0$ (using the Lemma 2.2.13). So, we get that $g - f = Hf$ where H is the matrix which components are $H_{i,j}(x)$. Thus, $H \in M_n(\mathfrak{m})$. Since $Id + H$ is invertible (1.3.1), we get that $g = (Id + H)f$, which proves the $\mathbb{C}$-A-equivalence. □

Definition 2.2.16. Let $m = (m_1, \ldots, m_n) \in (\mathbb{N}_0)^n$. The m-Pham map is $\Phi^m : \mathbb{C}^n \to \mathbb{C}^n$ such that $\Phi^m(z_1, \ldots, z_n) = (z_1^{m_1}, \ldots, z_n^{m_n})$.

Remark 2.2.17. Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a map-germ of multiplicity μ at 0. If we let $m = (\mu + 1, \ldots, \mu + 1)$, then the map-germ f is $\mathbb{C}$-A-equivalent at 0 to the germ $\Phi^m + \varepsilon f$ for all $\varepsilon \in \mathbb{R} - \{0\}$. This is because their difference is Φ^m, and they belong to $\mathfrak{m}^{\mu+1}$, so we apply 2.2.15.

Proposition 2.2.18. $\mathrm{ind}_0(\Phi^m) = \mu_0(\Phi^m)$.

Proof. If $m = (m_1, \ldots, m_n)$, we can compute both numbers separately. The index is equal to the number of roots of the map $\Phi^m - (\varepsilon_1, \ldots, \varepsilon_n)$ for a sufficiently small regular value $(\varepsilon_1, \ldots, \varepsilon_n)$ of Φ^m 2.2.9. The system $x_i = \varepsilon_i$ for $1 \leq i \leq n$ has $\prod_{i=1}^n m_i$ solutions if $\varepsilon_i \neq 0$ for every i.

On the other hand, $Q_{\Phi^m} = \mathbb{C}[x_1, \ldots, x_n]/\langle x_1^{m_1}, \ldots, x_n^{m_n} \rangle$ has a basis of monomials formed by the elements $\prod_{i=1}^n x_i^{k_i}$ where $0 \leq k_i < m_i$ for every i. So, the dimension of this algebra is $\prod_{i=1}^n m_i$. □

Definition 2.2.19. Let $g : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function and U an open set, $Hol(U)$ the algebra of holomorphic functions in U and $I_g(U) = \langle g_1, \ldots, g_n \rangle \subseteq Hol(U)$. The quotient $Q_g(U) = Hol(U)/I_g(U)$ is the local algebra of g on the domain U. The image of the polynomials in this quotient is called the *polynomial subalgebra*, and is denoted $Q_g[U]$. If $a_1, \ldots, a_m$ are the zeros of g in U, the *multilocal algebra* of g in U is the space $ML_g(U) = \oplus_{i=1}^m Q_{g,a_i}$, that is, the direct sum of the local algebras of the germs of g at the points a_i.

Lemma 2.2.20. *Suppose that the $\mathbb{C}$ dimension of the polynomial subalgebra of a map g in U is $\mu < \infty$. Then every zero of the map g is of finite multiplicity.*

Proof. We make an argument similar to the one in 2.2.13. If a is a zero of the map g and $\varphi_1, \ldots, \varphi_\mu$ are linear functions vanishing at a, then the images of the $\mu + 1$ polynomials $1, \varphi_1, \varphi_1\varphi_2, \ldots, \varphi_1 \cdots \varphi_\mu$ are linearly dependent. Thus, arguing as in 2.2.13, we find that there exists a function $h \in Hol(U)$ such that $h(a) \neq 0$ and $h\varphi_1 \cdots \varphi_\mu \in I_g(U)$. This says that $\varphi_1 \cdots \varphi_\mu \in I_{g,a}$, after inverting h (in the local algebra, not in the polynomial subalgebra). □

Proposition 2.2.21. *Let f_ε be a deformation of f. Then, for sufficiently small ε,*

$$|\{f_\varepsilon = 0\}| \leq \mu(f).$$

Proof. We name $f_\varepsilon(x) = (f_1(x, \varepsilon), \ldots, f_n(x, \varepsilon))$ and let $e_1, \ldots, e_\mu$ the polynomial generators of Q_f. Using the Thom-Martinet Preparation Theorem 1.3.5, any polynomial $P(x)$ can be decomposed in the form

$$P(x) = \sum_{j=1}^{\mu} g_j(\varepsilon) e_j(x) + \sum_{i=1}^{n} \varphi_i(x, \varepsilon) f_i(x, \varepsilon) \tag{2.1}$$

where each of the φ_i, g_j are holomorphic. The problem is that the domains of this functions can depend on the choice of the polynomial. However, we can choose domains not depending on the polynomial. We intersect the domains of the functions corresponding to the Thom-Martinet preparation theorem decomposition of $1, x_1, \ldots, x_n$ and $x_j e_k, 1 \leq j \leq n, 1 \leq k \leq \mu$. Since every polynomial of degree d can be put in the form $P = \sum_{j=1}^{n} x_j Q_j + c \cdot 1$ with $\deg(Q_j) < d$, we can apply induction with respect to $\deg(P)$ and obtain the desired representation in a fixed domain (that is, independent from the choice of the polynomial).

Thus, we can assume that $x \in U$, $\varepsilon \in V$ where U, V are fixed domains of the origin. Also, we can assume that all the zeros of $F_\varepsilon, \varepsilon \in V$ bifurcating from the origin lie in U (by reducing V). From our previous decomposition 2.1, we get that $\dim_{\mathbb{C}} Q_{f_\varepsilon}[U] \leq \mu$.

Let $a_1, \ldots, a_\nu$ be the roots of f_ε in U and $\Pi : Q_{f_\varepsilon}[U] \to ML_{f_\varepsilon}(U)$ be the natural map that sends a polynomial to its classes in the local algebra. The fact that this map is surjective implies the inequality. And this holds because given finite jets at the points a_i, there exist a polynomial having those jets at each a_i (this is Hermite interpolation). □

In the middle of the proof of this proposition we deduced that

Corolary 2.2.22. *$\sum \mu_{a_i}(F_\varepsilon) \leq \mu(f)$.*

And also

Corolary 2.2.23. *$\mathrm{ind}_0(f) \leq \mu(f)$.*

Proof. Applying our proposition to the deformation $F_\varepsilon = f - \varepsilon$, for ε a sufficiently small regular value of f, we get that $\mathrm{ind}_0(f) = |\{f = \varepsilon\}| \leq \mu$, by using 2.2.9 in the first equality and the proposition in the second one. □

Theorem 2.2.24. *The index of a holomorphic germ of finite multiplicity is equal to its multiplicity.*

Proof. If the map-germ does not vanish in the origin, then both the multiplicity and the index are obviously equal to 0. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a map-germ of finite multiplicity. By 2.2.17, we can choose a Pham map Φ such that f and $\Phi_\varepsilon := \Phi + \varepsilon f$ are $\mathbb{C}$-A-equivalent for $\varepsilon \neq 0$. Then, we choose a sufficiently small neighborhood U of 0 and a small ε. If we call a_i the roots of Φ_ε in U, we obtain the chain of inequalities

$$\mu_0(\Phi) \geq \sum \mu_{a_i}(\Phi_\varepsilon) \text{ by 2.2.22}$$
$$\mu_{a_i}(\Phi) \geq \mathrm{ind}_{a_i}(\Phi_\varepsilon) \text{ by 2.2.23}$$
$$\sum \mathrm{ind}_{a_i}(\Phi_\varepsilon) = \mathrm{ind}_0(\Phi) \text{ applying the Proposition 2.2.10 to the deformation } \Phi + tf$$
$$\mathrm{ind}_0(\Phi) = \mu_0(\Phi) \text{ because of Proposition 2.2.18.}$$

This chain of inequalities implies that $\mu_{a_i}(\Phi_\varepsilon) = \mathrm{ind}_{a_i}(\Phi_\varepsilon)$ for every a_i root of Φ_ε. Since $f(0) = 0$, then 0 is a root of Φ_ε, and therefore $\mu_0(\Phi_\varepsilon) = \mathrm{ind}_0(\Phi_\varepsilon)$. But since f and Φ_ε are $\mathbb{C}$-A-equivalent, we know that

$$\mu_0(f) = \mu_0(\Phi_\varepsilon) \text{ because of 2.2.12}$$
$$\mathrm{ind}_0(f) = \mathrm{ind}_0(\Phi_\varepsilon) \text{ because of 2.2.8.}$$

So, this implies that $\mu_0(f) = \mathrm{ind}_0(f)$. □

Corolary 2.2.25. *An isolated root of* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *has finite multiplicity.*

Proof. We would like to use our Theorem 2.2.24, to say that the multiplicity of that root is exactly its index; the problem is that we cannot use it directly on f since we don't know if it has finite multiplicity (moreover, that is what we would like to prove). The idea is to build another function germ g of finite multiplicity that "looks like" f (in the sense that they have the same index and are $\mathbb{C}$-A-equivalent). Let k be the number $\operatorname{ind}_0(f)$ and $g = f_{l-1} + \varepsilon\Phi^{(l,\dots,l)}$ where $l > k+1$, f_{l-1} is the taylor polynomial of f of degree $l-1$ and $\Phi^{(l,\dots,l)}$ is a Pham map (defined in 2.2.16). It is clear that the $k+1$-jets of f and g at 0 are equal. Also, the germ g is of finite multiplicity. Indeed, in the local algebra, the relation $\varepsilon\Phi^{(l,\dots,l)} = -f_{l-1}$ allows us to reduce the degree of each polynomial if its degree in one of the variables is greater or equal than l. This says that the subalgebra of $Q_{\nabla(g)}$ generated by polynomials is of finite dimension. This implies that $Q_{\nabla(g)}$ is finitely generated, by 2.2.20. If we define a ball B in the domain of convergence of the germ of f at 0 such that f vanishes only in the origin, we can choose l and ε such that $||f|| > ||f-g||$ in ∂B.

Finally, since for $0 \le t \le 1$, we get that $||tg + (1-t)f|| = ||f + t(g-f)|| \ge ||f|| - t||f-g|| > (1-t)||f-g|| \ge 0$, the maps $\frac{f}{||f||}$ and $\frac{g}{||g||}$ are homotopic, through the homotopy $\frac{tg+(1-t)f}{||tg+(1-t)f||}$. Therefore, $\operatorname{ind}_0(g) \le \deg(\frac{g}{||g||} : \partial B \to S_1(0)) = \operatorname{ind}_0(f) = k$. Since g has finite multiplicity at 0, we know that $\mu_0(g) = \operatorname{ind}_0(g) \le k$ because of the Theorem 2.2.24. Since $f - g \in \mathfrak{m}^{\mu_0(g)+1}$, the germs at 0 of f and g are $\mathbb{C}$-A-equivalent because of 2.2.15 and thus have the same (finite) multiplicity at 0 (this is because of 2.2.12). □

2.3 Tougeron's finite determinacy theorem

To classify critical points, it is necessary to describe the action of the infinite-dimensional Lie group of diffeomorphism-germs over the infinite-dimensional space of map-germs. The Tougeron's theorem states that any function-germ at an isolated critical point (or equivalently, of finite multiplicity, as seen in the last chapter) is equivalent to a polynomial. This helps us reduce the description of isolated singularities to the action over a finite-dimensional space of map-germs.

To prove Tougeron's theorem, we are going to introduce a method proposed by Thom, which is called the *homotopy method*. Say that (in a more general way and in the real case) we want to have a left-right equivalence between two maps between differentiable manifolds M and N, say $f, g : M \to N$, so we want to find two diffeomorphisms H and K such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow H & & \downarrow K \\ M & \xrightarrow{g} & N \end{array} .$$

To find those diffeomorphisms, we find a homotopy $F : M \times I \to N$ that joins f and g and try to decompose the previous commutative diagram into many "infinitesimal" ones, by trying to find two diffeomorphisms

$$\begin{array}{ccc} M & \xrightarrow{F_t} & N \\ \downarrow{\scriptstyle H_{\Delta t}} & & \downarrow{\scriptstyle K_{\Delta t}} \\ M & \xrightarrow{F_{t+\Delta t}} & N \end{array}$$

for small Δt.

In the rest of this subsection, let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $\mathfrak{m}$ the maximal ideal, $\varphi \in \mathfrak{m}^{\mu+2}$, where $0 < \mu < \infty$ is the multiplicity of the map $\nabla(f)$ at the point 0 and $f_j, \varphi_j : \mathbb{C}^n \to \mathbb{C}$ the maps $\partial_j f$ and $\partial_j \varphi$ for $1 \leq j \leq n$.

Lemma 2.3.1. *1. Every monomial of sufficiently high degree ($\geq \mu$) belongs to the ideal $I_{\nabla(f+t\varphi)}$, that is $\mathfrak{m}^\mu \subseteq I_{\nabla(f+t\varphi)}$ with $t \in [0,1]$ a constant.*

2. The homological equation with unknown v_t a vector depending on t

$$v_t \cdot (f + t\varphi) = \alpha$$

is solvable for every $t \in [0,1]$ if α is a monomial of degree $\mu+1$. The $\cdot$ denotes diferentiation in the direction of the vector v_t. Moreover, the solution depends smoothly on t and vanishes at the origin.

Proof. 1. There is a finite number (which we call r) of monomials of degree μ, which are $\{M_j\}_{j=1}^r$. By means of Lemma 2.2.13 M_i is one of those monomials. We have that $M_i \in I_f$ and

$$M_i = \sum_{j=1}^n f_j h_{j,i}(x) = \sum_{j=1}^n (f_j + t\varphi_j) h_{j,i}(x) - \sum_{j=1}^n t\varphi_j h_{j,i}(x),\ h_{j,i} \in \mathcal{O}_n.$$

Since $\varphi_j \in \mathfrak{m}^{\mu+1}$, the term substracted in the right hand side can be written as a linear combination of monomials of degree μ with coefficients in $\mathfrak{m}$. Then, we have

$$M_i = \sum_{j=1}^n (f_j + t\varphi_j) h_{j,i}(x) - \sum_{j=1}^r M_j \left(\sum_{s=1}^n t x_s a_{j,s,i}(x) \right),\ a_{j,s} \in \mathcal{O}_n.$$

By taking the same decomposition for all the monomials of degree μ, we get a system of equations
$$\begin{cases} M_1 + \sum_{j=1}^r M_j \left(\sum_{s=1}^n t x_s a_{j,s,1}(x) \right) & = \sum_{j=1}^n (f_j + t\varphi_j) h_{j,1}(x) \\ & \vdots \\ M_r + \sum_{j=1}^r M_j \left(\sum_{s=1}^n t x_s a_{j,s,r}(x) \right) & = \sum_{j=1}^n (f_j + t\varphi_j) h_{j,r}(x) \end{cases}$$
which is a system of equations of the form $(\mathrm{Id} + A_t)M = B$. In the equation, $B = \left(\sum_{j=1}^n (f_j + t\varphi_j) h_{j,i}(x) \right)_{i=1}^r$ and $M = (M_1, \ldots, M_r)$ are r-dimensional vectors and $A_t = \left(\sum_{s=1}^n t x_s a_{j,s,i}(x) \right)_{i,j}$ is a matrix with coefficients in $\mathfrak{m}$. If we know $(\mathrm{Id} + A_t)$ is invertible,

then $M = (\mathrm{Id} + A_t)^{-1}B$ and as B has coefficients in $I_{\nabla(f+t\varphi)}$, we conclude that M does too. And this has been done in 1.3.1.

2. If we fix $t \in [0,1]$ and write $\alpha = x_i M_j$ where M_j is a monomial of degree μ we know there exists a solution of the equation

$$v_t \cdot (f + t\varphi) = M_j$$

because $M_j \in I_{\nabla(f+t\varphi)}$. Also, using the same notation that we used in the proof of the first part of this lemma, as $(\mathrm{Id} + A_t)^{-1} = \mathrm{adj}(\mathrm{Id} + A_t)\det(\mathrm{Id} + A_t)^{-1}$ (having in mind that $\det(\mathrm{Id} + A_t)^{-1}$ is a unit), we can see that the solution depends smoothly of $t \in [0,1]$. By multiplying by x_i we get the desired solution v_t.

□

Definition 2.3.2. We say that a k-jet is *sufficient* if any two functions with that k-jet are equivalent.

Theorem 2.3.3 (Tougeron)**.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an holomorphic map-germ at a critical point of finite multiplicity μ. Then its $\mu + 1$-jet is sufficient.*

Proof. We join f and $f + \varphi$ with the homotopy $F(x,t) = (f + t\varphi)(x)$ and we look for a family of local diffeomorphisms g_t such that

$$\begin{cases} (f + t\varphi)(g_t(x)) \equiv f(x) \\ g_0(x) \equiv x \\ g_t(0) \equiv 0 \end{cases}.$$

If we take $\frac{d}{dt}$ in both sides of the equation, and we call $v_s(g_s(x)) = \left(\frac{d}{dt} g_t(x)\right)|_{t=s}$ we get another equation

$$\varphi(g_t(x)) + v_t(g_t(x)) \cdot (f + t\varphi)(g_t(x)) \equiv 0. \tag{2.2}$$

As $\alpha = -\varphi \in \mathfrak{m}^{\mu+2}$, we know that there exists a decomposition $\alpha(x) = \sum_{i=1}^{s} \alpha_i(x) c_i(x)$ where the α_i are monomials of degree $\mu + 1$ and $c_i \in \mathfrak{m}$. We can solve for every i the homological equations with unknown (v_i)

$$\begin{cases} (v_i)_t \cdot (f + t\varphi) \equiv \alpha_i \\ (v_i)_t(0) \equiv 0 \end{cases}.$$

Which means that $v_t = \sum_{i=1}^{s} c_i(x)(v_i)_t$ is a solution for 2.2. To find g_t, we use the fact that

$$\begin{cases} \frac{d}{dt} g_t(x) \equiv v_t(g_t(x)) \\ g_t(0) \equiv 0 \\ g_0(x) \equiv f(x) \end{cases}$$

is an ordinary differential equation for g_t and its (only) solution solves 2.2. By integrating the equation in t between 0 and s we get that

$$(f + t\varphi)(g_t(x))|_{t=0}^{t=s} \equiv (f + s\varphi)(g_s(x)) - f(x) \equiv 0$$

which means that g_1 is a right change of coordinates between f and $f + \varphi$. □

Remark 2.3.4. This in particular says that every isolated singularity has a polynomial representative of degree less than $\mu + 1$, where μ is its multiplicity in the critical point. Thus, this simplifies the classification of isolated singularities of a certain multiplicity to the classification of orbits of an infinite dimensional group action over a finite-dimensional manifold (the $\mu + 1$-jets).

Remark 2.3.5. The particular case of $\mu = 1$, Tougeron's theorem says that every critical point of index 1 (that is, non-degenerate critical point by 2.1.8) is equivalent to its 2 jet, which can be made equivalent, by completing the square several times, to a sum of variables squared; this is exactly the Morse lemma 1.2.4.

Example 2.3.6 (Whitney)**.** Let us consider the holomorphic function in three variables $f(x, y, z) = xy(x + y)(x - zy)(x - e^z y)$. Each plane $z = c, c \in \mathbb{C}$ fixed, intersects the set $\{f(x, y, z) = 0\}$ along 5 curves that intersect in the point $(0, 0, c)$. The cross-ratios of the tangents of 4 of those 5 curves depend on the plane $z = c$ chosen. One can check that this dependance is not algebraic (because of the factor e^z appearing in the expression of f), and this proves that this function is not equivalent to a polynomial (in that case, the dependance must be algebraic).

Chapter 3

Versal deformations

Generally, when we consider the set of all singularities, the main interest is the study of the nondegenerate critcal points, since they appear generically; that is, we may get rid of complicated singularities by small perturbations (that is the content of the Transversality theorems, see for example [3] Chapter 2, [5] Chapter 3). However, in many cases we are not interested in the study of an individual object, but in a family of objects, depending on parameters. In this case, degenerate singularities can be "irremovable". Take for instance, the case of $x^3 + tx$. It has a degenerate singularity for $t = 0$ and every sufficiently close family will have a degenerate critical point for t close to 0, although for each fixed value of the parameter the singularity is removable by a generic perturbation of the map. Therefore, the natural object of study is not the degenerate singularity, but the family in which this singularity becomes irremovable; this will be the main topic of this chapter.

Definition 3.0.1. Let G be a Lie group acting on a mainfold M and $f \in M$. A *deformation* of f is a smooth map-germ F from a manifold Λ (called the *base*) to M at a point $0 \in \Lambda$ for which $F(0) = f$.

Two deformations $F, F' : \Lambda \to M$ are said to be *equivalent* if they have the same base and there exists a deformation $g : \Lambda \to G$ of the element $1 \in G$ such that

$$F'(\lambda) = g(\lambda)F(\lambda); \lambda \in \Lambda.$$

Definition 3.0.2. If $\theta : (\Lambda', 0) \to (\Lambda, 0)$ is a smooth map, and $F : (\Lambda, 0) \to (M, f)$ is a deformation, we call $\theta^*F = F \circ \theta$ the *deformation induced from* F *by* θ.

A deformation F is *versal* if every deformation of f is equivalent to one induced from F.

A deformation F of f is *miniversal* if it is versal and if the dimension of its base is less or equal than the dimension of the base of any other versal deformation of f (that is, the dimension of its base takes its least possible value).

Example 3.0.3. The identity map $Id : M \to M$ is always a versal deformation, but is not in general a mini-versal deformation. Since we want to parametrize the space of functions in the simpler way, the definition of versal deformation is not enough; mini-versal deformations will be the way to understand a neighborhood of a point in the space of functions.

Remark 3.0.4. The transversality of the deformation $F : (\Lambda, 0) \to (M, f)$ to the orbit Gf of f is a necessary condition for the versality of F. In fact, suppose that

$$F_* T_0 \Lambda + T_f Gf \subsetneq T_f M.$$

If we take an equivalent deformation to F, say H, then $F_* T_0 \Lambda + T_f Gf = H_* T_0 \Lambda + T_f Gf$ holds. And if we take any induced deformation $\theta^* F$, we know that $(\theta^* F)_* T_0 \Lambda + T_f Gf \subseteq F_* T_0 \Lambda + T_f Gf$. This says that every deformation F' of f satisfies

$$F'_* T_0 \Lambda + T_f Gf \subsetneq T_f M$$

which is obviously not true (take F' the identity map of M).

Theorem 3.0.5. *A minimal transversal to Gf at f in M is a miniversal deformation of f.*

Proof. Let $F : (\Lambda, 0) \to (M, f)$ be a minimal transversal deformation and let K be a transversal to the stabiliser $E_G(f)$ of f. The product operation $p : K \times F(\Lambda) \to M$ defines a smooth map-germ at $(1, 0)$ that is also a diffeomorphism-germ. This is because the differential is suryective (as a consequence of the previous remark) and also injective, or we can reduce the dimension of Λ and keep it a transversal deformation. By the inverse function theorem, the product is a diffeomorphism.

Then, let $F' : (\Lambda', 0) \to (M, f)$ be another deformation of f. For all $\lambda' \in \Lambda'$, we have that $p^{-1}(F'(\lambda')) = (\beta(\lambda'), \gamma(\lambda')) \in K \times F(\Lambda)$. Finally, $F'(\lambda') = \beta(\lambda') \cdot \gamma(\lambda') = \beta(\lambda') \cdot F(F^{-1}(\gamma(\lambda')))$.

□

Definition 3.0.6. The *modality* of the point $f \in M$ under the action of a Lie group G is the least number m such that a small neighborhood of f is covered by a finite number of m-parameter families of orbits.

Example 3.0.7. Let M be the manifold of quadruples of lines passing through the origin in $\mathbb{C}^3$ and G be the group $GL(3, \mathbb{C})$. G acts on M by multiplication. Let us describe the orbits of this action.

Firstly, the quadruples of lines that are not contained in a common plane are one orbit. Indeed, if we multiply three linearly independent vectors on $\mathbb{C}^3$ by the same invertible matrix, we obtain again three linearly independent vectors. Also, if v_1, v_2, v_3, v_4 and w_1, w_2, w_3, w_4 are a pair of quadruples of vectors that direct two non-coplanar quadruples of lines, then there is a matrix g such that $gv_i = w_i$ for $i = 1, 2, 3$. Then, if $v_4 = a_1 v_1 + a_2 v_2 + a_3 v_3$ and $w_4 = b_1 w_1 + b_2 w_2 + b_3 w_3$, the matrix $\tilde{g}$ such that $\tilde{g} v_i = \frac{b_i}{a_i} w_i$ if $a_i \neq 0$ and $\tilde{g} v_i = w_i$ if $a_i = 0$ sends one quadruple into the other. This orbit is a 0-parametric family.

In the case of the quadruples of lines that lie in a common plane, there is a numerical invariant of the action: the cross-ratios of the four lines. So, we will not be able to cover a neighborhood of any of these points with finitely many orbits; we will need at least a uniparametric family. Since every quadruple of coplanar lines with a fixed cross-ratio can be moved to any other quadruple with the same cross ratio by the action of an element of the group, we can cover a neighborhood of any coplanar quadruples of lines with a uniparametric family that varies the cross ratios and the orbit (or 0-parametric family) of non-coplanar quadruples of lines. Thus, the modality is 1.

We have introduced this deformations in the finite-dimensional case to generalize its study to our infinite dimensional case, in which we have a space M of maps (smooth or holomorphic) and an infinite dimensional Lie group, which is the group of changes of variables, right-acting.

Definition 3.0.8. A deformation with base $\Lambda = \mathbb{C}^l$ of the map-germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is the germ at zero of a map $F : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to \mathbb{C}$ such that $F(x, 0) = f(x)$.

A deformation F' is equivalent to F if $F'(x, \lambda) = F(g(x, \lambda), \lambda)$ where $g(x, 0) \equiv x$ and $g : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to (\mathbb{C}^n, 0)$ is a holomorphic map-germ.

A deformation G is induced from F by θ if

$$G(x, \lambda') = F(x, \theta(\lambda'))$$

with $\theta : (\mathbb{C}^{l'}, 0) \to (\mathbb{C}^l, 0)$ is an holomorphic map-germ.

A deformation $F : (\mathbb{C}^n \times \mathbb{C}^l, 0) \to \mathbb{C}$ of the germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is said to be versal if every deformation of f is equivalent to a deformation induced from F. Equivalently, if every deformation F' of f can be represented as

$$\begin{cases} F'(x, \lambda') = F(g(x, \lambda'), \theta(\lambda')) \\ \quad g(x, 0) \equiv x \\ \quad \theta(0) = 0 \end{cases}.$$

If we have a versal deformation F of the germ f, we know that it is transversal to the orbit of f (using the 3.0.4). In the finite dimensional case, this is also a sufficient condition as we already proved. This also holds in the infinite-dimensional case of the action of biholomorphic changes we just defined. But in this case, we have to define a notion of transversality in the infinite-dimensional case.

Remark 3.0.9. Let us assume that F is a versal deformation of $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. Then, there exist a family of holomorphic map-germs $g : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ and a holomorphic function $\theta : (\mathbb{C}, 0) \to (\mathbb{C}^l, 0)$ such that

$$f(x) + \lambda' \alpha(x) = F(g(x, \lambda'), \theta(\lambda'))$$

where $\alpha : (\mathbb{C}^n, 0) \to \mathbb{C}$ is any map. Taking $\frac{d}{d\lambda'}|_{\lambda'=0}$ both sides of the equality, we get

$$\alpha(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \frac{\partial g_i}{\partial \lambda'}(x, 0) + \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i}(x, 0) \frac{\partial \theta_i}{\partial \lambda'}(0).$$

So, the infinitesimal condition should be that any function can be written as a sum of an element of $I_{\nabla(f)}$ (which is an element of the first summation, and takes the role played by the tangent to the orbit in the finite dimensional case) and a linear combination of the partial derivatives $\frac{\partial F}{\partial \lambda_i}|_{\lambda'=0}$.

Definition 3.0.10. A deformation $F(x, \lambda)$ of the germ $f(x)$ is called *infinitesimally versal* if every function-germ $g(x)$ can be written as

$$g(x) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i} + \sum_{j=1}^{l} c_j \frac{\partial F}{\partial \lambda_j}|_{\lambda=0}$$

where $h_i(x)$ are holomorphic map-germs and c_j are constants. That means, if $\frac{\partial F}{\partial \lambda_j}|_{\lambda=0}$ generate the local algebra $Q_{\nabla f}$ as a $\mathbb{C}$-module.

Theorem 3.0.11. *Any infinitesimally versal deformation of a function-germ is versal.*

Proof. Let $F(x,\lambda), \lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^k, 0)$ be an infinitesimally versal deformation of a germ f and let $F'(x,\lambda'), \lambda' = (\lambda'_1, \dots, \lambda'_k) \in (\mathbb{C}^{k'}, 0)$ be another deformation of f. First, consider the deformation $\tilde{F}(x,\lambda,\lambda') = F(x,\lambda) + F'(x,\lambda') - f(x)$. It is both a deformation of f with parameters (λ, λ') and a deformation of F with parameters λ'. Thus, it is an infinitesimally versal deformation of f. It is clear that F' is induced from $\tilde{F}$ (replacing $\lambda = 0$). So, we will manage to make $\tilde{F}$ equivalent to a deformation induced from F. Moreover, it is enough to prove it for $k' = 1$: in this case, we make step by step $\tilde{F}(x,\lambda,\lambda'_1,\dots,\lambda'_s,0,\dots,0)$ equivalent to one induced from $\tilde{F}(x,\lambda,\lambda'_1,\dots,\lambda'_{s+1},0,\dots,0)$ for every $1 \le s \le k'-1$ and finally $\tilde{F}(x,\lambda,\lambda'_1,0,\dots,0)$ equivalent to one induced from $\tilde{F}(x,\lambda,0) = F(x,\lambda)$.

Now, we reduced our problem to prove that the deformation $\Phi(x,\lambda,\lambda'), \lambda \in \mathbb{C}^k, \lambda' \in \mathbb{C}; \Phi(x,\lambda,0) = F(x,\lambda)$ is equivalent to one induced from F. This means that we have to find $\theta : (\mathbb{C}^{k+1}, 0) \to \mathbb{C}^k, g' : (\mathbb{C}^{n+k+1}, 0) \to \mathbb{C}^n$ such that

$$\Phi(x,\lambda,\mu) = F(g'(x,\lambda,\mu), \theta(\lambda,\mu)).$$

We can think $h'_\mu(x,\lambda) = (g'(x,\lambda,\mu), \theta(\lambda,\mu))$ as a 1-parameter family of (local) biholomorphisms. Thus, by applying $(h')^{-1}_\mu$ both sides of the equation, we need to find $\varphi : (\mathbb{C}^{k+1}, 0) \to \mathbb{C}^k, g : (\mathbb{C}^{n+k+1}, 0) \to \mathbb{C}^n$ such that

$$\Phi(g(x,\lambda,\mu), \varphi(\lambda,\mu), \mu) = F(g'(x,\lambda,\mu), \theta(\lambda,\mu)) \tag{3.1}$$

where $h_\mu(x,\lambda) = g(x,\lambda,\mu), \varphi(\lambda,\mu)$ is a 1-parameter family of local biholomorphisms. Since we have an equation holding and we have to find a 1-parameter family of local biholomorphisms, we can use the *homotopy method*, formerly described for the proof of Tougeron's finite determinacy theorem 2.3.3. In this direction, we consider the vector field V_μ corresponding to the family h_μ and depending on μ defined by the equation

$$V_\mu \circ h_\mu = \frac{\partial h_\mu}{\partial \mu}. \tag{3.2}$$

Thus, we have an expression for V_μ of the form

$$V_\mu = \sum_{i=1}^{n} H_i(x,\lambda,\mu)\frac{\partial}{\partial x_i} + \sum_{j=1}^{k} \xi_j(\lambda,\mu)\frac{\partial}{\partial \lambda_j}$$

where $H_i, \xi_j, 1 \le i \le n, 1 \le j \le k$ are holomorphic function-germs.

Thus, after taking $\frac{\partial}{\partial \mu}$ both sides of the equality in 3.1, we get

$$\frac{\partial \Phi}{\partial \mu} + \sum_{i=1}^{n} H_i(x,\lambda,\mu)\frac{\partial \Phi}{\partial x_i} + \sum_{j=1}^{k} \xi_j(\lambda,\mu)\frac{\partial \Phi}{\partial \lambda_j} \equiv 0. \tag{3.3}$$

If we manage to solve the equation

$$\sum_{i=1}^{n} H_i(x,\lambda;\mu)\frac{\partial\Phi}{\partial x_i} + \sum_{j=1}^{k} \xi_j(\lambda;\mu)\frac{\partial\Phi}{\partial\lambda_j} \equiv \alpha(x,\lambda;\mu) \tag{3.4}$$

with unknowns H_i, ξ_j, we can recover the family h_μ from V_μ in the 3.2 and integrate 3.3 to show that this h_μ satisfies the required relation. Indeed, the hypothesis of infintiesimal versality of F (that implies the infinitesimal versality of Φ), say that the equation has a solution for $\lambda = 0, \mu = 0$, so we need a tool that allows us to "extend" this solution along the evolution of parameters. This tool is exactly the Thom-Martinet Preparation Theorem 1.3.5. Indeed, if we put $y = (\lambda;\mu), I = \langle \frac{\partial\Phi}{\partial x_1}, \dots, \frac{\partial\Phi}{\partial x_n}\rangle, e_i = \frac{\partial\Phi}{\partial\lambda_i}$, the thesis of its theorem says exactly that 3.4 has a solution in the class of germs of analytic functions. Thus, the theorem is proved.

□

Remark 3.0.12. The proof gives us a clearer interpretation of the Thom-Martinet Preparation Theorem: it is a theorem that "extends" the solution of an equation such as 3.4 (where the unknowns are the functions) "along the parameters".

Corolary 3.0.13. *The base of a miniversal deformation of a critical point of a map-germ f has dimension $\mu(f)$, its multiplicity at the point. Moreover, a miniversal deformation has the form*

$$f(x) + \sum_{j=1}^{\mu(f)} \lambda_j v_j$$

where the $\{v_1, \dots, v_{\mu(f)}\}$ is a basis of $Q_{\nabla f}$.

Proof. To be able to generate $Q_{\nabla f}$, its dimension must be at least $\mu(f)$. If $\{v_1, \dots, v_{\mu(f)}\}$ is a basis of $Q_{\nabla f}$, then we have that $f(x) + \sum_{j=1}^{\mu(f)} \lambda_j v_j$ is an infinitesimally versal deformation of f and therefore is a miniversal deformation. □

Theorem 3.0.14 (Uniqueness of miniversal deformations)**.** *Any miniversal deformation F of a germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is equivalent to a deformation induced from any other miniversal deformation F' by a biholomorphism of their bases.*

Proof. Let k be the dimension of the base of a miniversal deformation, that is $k = \mu(f)$. Then, we know there exist $g : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^n, 0), \theta : (\mathbb{C}^k, 0) \to (\mathbb{C}^n, 0)$ such that

$$\begin{cases} F'(x,\lambda) = F(g(x,\lambda), \theta(\lambda)) \\ g(x,0) \equiv x \\ \theta(0) = 0 \end{cases} .$$

We have to prove that $D\theta(0)$ is an invertible matrix. Now, taking $\frac{\partial}{\partial\lambda_i}|_{\lambda=0}$ both sides of the equation, we get

$$\frac{\partial F'}{\partial \lambda_i}(x,0) = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(x,0)\frac{\partial g_j}{\partial \lambda_i} + \sum_{s=1}^{k} \frac{\partial F}{\partial \lambda_s}(x,0)\frac{\partial \theta_s}{\partial \lambda_i}.$$

As $\frac{\partial F}{\partial x_j}(x,0) = \frac{\partial f}{\partial x_i}(x)$, we get

$$\frac{\partial F'}{\partial \lambda_i}(x,0) = \frac{\partial F}{\partial \lambda}(x,0) \cdot \frac{\partial \theta}{\partial \lambda_i}(0)$$

in the local algebra $Q_{\nabla(f)}$. Assume that there exist constants $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $\sum_{i=1}^{k} \alpha_i \frac{\partial \theta}{\partial \lambda_i}(0) = 0$. We have to prove that $\alpha_i = 0$ for every i (this says that the rows of $D\theta(0)$ are linearly independent). In fact,

$$\sum_{i=1}^{k} \alpha_i \frac{\partial F'}{\partial \lambda_i}(x,0) = \sum_{i=1}^{k} \frac{\partial F}{\partial \lambda}(x,0) \cdot (\alpha_i \frac{\partial \theta}{\partial \lambda_i}(0)) = 0$$

holds in $Q_{\nabla(f)}$, and since $\left\{ \frac{\partial F'}{\partial \lambda_i}(x,0) \right\}_{i=1}^{n}$ is a linearly independent set, then $\alpha_i = 0$ for every i. □

Definition 3.0.15. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a function-germ (that is $f \in \mathfrak{m}$ the maximal ideal in the space of function-germs). The *modality* m of this function-germ $f(x)$ is the modality of any of its jets $j^k(f)$ (in $\mathfrak{m}$) for every $k \geq \mu(f) + 1$. This is well defined because of Tougeron's theorem 2.3.3. The function-germs of modalities 0,1 and 2 respectively are called *simple*, *unimodal* and *bimodal* respectively. We will give the complete classification (and by this, we mean a list of normal forms) of simple singularities in the last chapter and explain some methods (even though not all of them) that allow us to classify all the unimodal and bimodal singularities. This result is due to Arnold in [2]. The full list of unimodal and bimodal singularities can be found in [3] Chapter 16, or [1] Chapter 1, Section 2.3.

Remark 3.0.16. The definition of modality considers the smallest function space $\mathfrak{m}$; otherwise, all critical points will have modality greater than 0.

Example 3.0.17. The modality of the function-germ $f(x) = x^2$ is 0. This is an immediate consequence of the Morse Lemma.

To classify the singularities of map-germs, we will try to give normal forms, which are essentially some choice of a member of each orbit. This choice is not usually unique. Let us give a definition.

Definition 3.0.18.

- A *class of singularities* K is any subset of $\mathscr{O}_n$ that is invariant under the action of the group of biholomorphic map-germs.
- A *normal form* for a class of singularities K is a smooth map $\Phi : B \to M$ from a finite-dimensional vector space B into the set of polynomials M that satisfies

  1. $\Phi(B)$ intersects all orbits in K.
  2. The preimage of any orbit in K under Φ is a finite set.

3. The preimage of K^c under Φ is contained in some proper hypersurface in B.

- A normal form is said to be *polynomial* if Φ is a polynomial with coefficients in M (the set of polynomials) and it is said to be *simple* if $\Phi(b_1, \ldots, b_k) = p(x) + \sum_{i=1}^{k} b_i x^{m_i}$, where $p(x)$ is a fixed polynomial and $m_i \in \mathbb{N}$.

Remark 3.0.19. In this work, we will not check that our normal forms satisfy this definition (since we deal only with simple singularities, we do not need this complicated definition, that is useful when the classification is more complicated). Instead, we will think of normal forms as some (probably simultaneous) choice of a member from each orbit. This choice is not unique, so we must do it in a natural way.

Chapter 4

Quasihomogeneous singularities

In this section, we introduce and work with quasihomogeneous and semiquasihomogeneous morphisms with the idea of reducing quasihomogeneous and semiquasihomogeneous singularities to normal forms. In this chapter, we will mainly work with the algebra of polynomials, but they can be replaced by power series or germs (except it explicitly says that they cannot).

A motivating example for the study of semiquasihomogeneous functions is the fact that every function-germ in two variables with 3-jet equivalent to $x^2y + y^3$ can be made equivalent to its taylor polynomial of order 3, say its "principal part". This is of course not true in general: indeed, the initial part may have a non-isolated singularity, while the whole singularity may not (take for example $x^3 + y^5$ in two variables). The idea of working with quasihomogeneous and semi-quasihomogeneous functions, is that it allows us to generalize the idea of making a function equivalent to its principal part by relaxing the notion of principal part of a power series. Indeed, we will prove that (under this relaxed new notion of degree) if the principal part has an isolated singularity, then we can make it equivalent to a normal form, depending only on its principal part. This will be our main tool to reduce singularities to normal forms in the next (and last) chapter. The main source is [3], Part 2, Chapter 12.

From now on, if $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ and $k \in (\mathbb{Z}_{\geq 0})^n$, we denote $x^k := \prod_{i=1}^n x_i^{k_i}$.

4.1 The Newton diagram

Definition 4.1.1. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an holomorphic function-germ and let $\sum_{k\in(\mathbb{Z}_{\geq 0})^n} f_k x^k$ be its Taylor series. The *Newton support* of f is the set

$$\operatorname{supp}(f) = \{k \in (\mathbb{Z}_{\geq 0})^n : f_k \neq 0\}.$$

The *Newton polyhedron* is the convex hull of $\operatorname{supp}(f) + (\mathbb{Z}_{\geq 0})^n$ depicted in $(\mathbb{R}_{\geq 0})^n$. The *Newton diagram* of f is the union of compact faces of its Newton polyhedron.

Example 4.1.2. The Newton diagram and Newton polyhedron of the function $f(x, y) = x^6y + x^2y^2 + y^3$ are shown in the figure 4.1.

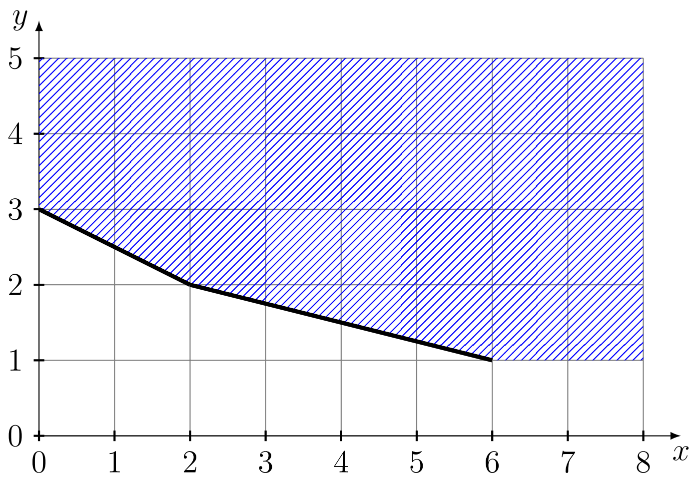


Figure 4.1: Newton diagram and polyhedron of $x^6y + x^2y^2 + y^3$.

Remark 4.1.3. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ. If we multiply f by any monomial x^k, its support gets transalted by the vector k, that has non-negative coordinates. Thus, given a Newton polyhedron Γ, the holomorphic function-germs whose supports are contained in Γ form an ideal in the ring $\mathscr{O}_n$ of holomorphic function-germs.

4.2 Quasihomogeneous functions

Definition 4.2.1. Consider the space $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$. An holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be *quasihomogeneous* of degree d and indices (or weights) $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n_{>0}$ if for every $\lambda > 0$, the equality

$$f(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, \ldots, x_n)$$

holds. If we write the Taylor series $f = \sum_k f_k x^k$, the condition means that all the indices of the non-null terms that appear in the series, belong to the hyperplane $\Gamma = \{k : \langle \alpha, k \rangle = d\}$. When $d = 1$, this space Γ is called the *diagonal*. If we divide all the weights by d, we can always assume that a quasihomogeneous function has degree 1. We also say that a monomial x^k has generalized degree d (fixing first α) if $\langle \alpha, k \rangle = d$. This gives a filtration in the ring of power series.

Definition 4.2.2. A quasihomogeneous function is *non-degenerate* if 0 is an isolated critical point (or equivalently, of finite multiplicity because of 2.2.24). They form an algebraic hypersurface in the linear space of quasihomogeneous polynomials.

Definition 4.2.3. A polynomial f has order d (we note it $\varphi(f)$) if all its monomials have degree d or higher. In this case, we call d the *quasi-degree* of f. We denote A_d the space of power series/germs/polynomials of order d and $A_{<d}$ the space of series/germs/polynomials of order greater than d. By convention, we say that $\varphi(0) = +\infty$.

Remark 4.2.4. $A_{d'} \subseteq A_d$ if $d < d'$, and $\varphi(f)$ is the biggest rational number d such that $f \in A_d$. Also, as $0 \in A_d$ and the order of a product of monomials is the sum of its orders, we have that A_d is an ideal in the algebra of polynomials A. This gives a filtration in the ring $\mathscr{O}_n$ (when $\alpha = (1, \dots, 1)$, this is the usual filtration by degree). By taking the quotient A/A_d, we are identifying polynomials (or maps, more in general) that have the same Taylor polynomial of degree d.

Definition 4.2.5. With the last definition in mind, we say that A/A_d is the *algebra of d quasi-jets* and we call its elements, d quasi-jets.

Definition 4.2.6. A power series/polynomial is said to be *semiquasihomogeneous* of degree d and weights $\alpha_1, \dots, \alpha_n$ if $f = f_0 + g$ where f_0 is a non-degenerate quasihomogeneous polynomial of degree d weights α and g is a polynomial of order greater than d.

Remark 4.2.7. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a quasihomogeneous function of degree d and type α. Then, its partial derivatives are also quasihomogeneous. Indeed, taking $\frac{\partial}{\partial x_i}$ on both sides of the expression

$$f(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) = \lambda^d f(x_1, \dots, x_n)$$

we obtain

$$\lambda^{\alpha_i} \frac{\partial f}{\partial x_i}(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) = \lambda^d \frac{\partial f}{\partial x_i}(x_1, \dots, x_n).$$

This says that $\frac{\partial f}{\partial x_i}$ is a quasihomogeneous function of degree $d - \alpha_i$ and weights α.

One of the main goals of the section will be to prove the next result.

Theorem 4.2.8. *Let f_0 be a non-degenerate quasihomogeneous function (or polynomial), and let us fix a basis of monomials of the local algebra of f_0. Let $e_1, \dots, e_s$ be the monomials of this basis whose indices lie strictly over the diagonal. Then every semiquasihomogeneous function with quasihomogeneous part f_0 is equivalent to a function $f_0 + \sum_{k=1}^{s} c_k e_k$ with c_k constants.*

First, let us show that a monomial basis for the local algebra of a quasihomogeneous holomorphic and non-degenerate function is also a basis for the local algebra of all semi-quasihomogeneous functions with such quasihomogeneous part.

Theorem 4.2.9. *If f is a semiquasihomogeneous function with quasihomogeneous part f_0, then $\mu(f) = \mu(f_0)$ in the point* 0.

Proof. We shall suppose that the degree of the quasihomogeneous part d is 1. Let us consider $S_t = \{x \in \mathbb{C}^n : |x_1|^{a_1} + \ldots + |x_n|^{a_n} = t\}$ where $a_i = \frac{1}{\alpha_i}$ the inverses of the weights, and remember that $\mu(f)$ is the degree of the map $\frac{\nabla(f)}{||\nabla(f)||}$ with source space S_t for small t. We also know that $T_t \circ S_1 = S_t$ where $T_t(x) = (t^{\alpha_1} x_1, \dots, t^{\alpha_n} x_n)$, so we can assume that the source space of $\frac{\nabla(f)}{||\nabla(f)||}$ is S_1. We also know that f_0 is nondegenerate, so at least one of the partial derivatives of f_0 is not zero. Therefore, $\max_{s=1,\dots,n} |\frac{\partial f_0}{\partial x_s}| \geq c > 0$.

On the other hand, we know that $\lambda f_0(x_1, \dots, x_n) = f_0(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n)$. If we take $\frac{\partial}{\partial x_s}$ on both sides of the equality, we get that $\frac{\partial f_0}{\partial x_s}$ is quasihomogeneous of degree $1 - \alpha_s$, and therefore $\frac{\partial f_0}{\partial x_s} \geq ct^{1-\alpha_s}$ for at least one $1 \leq s \leq n$ and for $x \in S_t$. Also, if $f = f_0 + f'$, where f' is a polynomial of order strictly greater than d, we know that $|\frac{\partial f'}{\partial x_s}| \leq Ct^{1+d-\alpha_s}$.

Using both inequalities, we can show that for a sufficiently small t, the map $f_0 + \theta f'$ has no critical points on the sphere S_t, for every $0 \leq \theta \leq 1$. This says that the degrees of the maps given by the gradients of f_0 and $f_0 + f'$ coincide.

□

Lemma 4.2.10. *Let F be a family of smooth functions that depend continuously on a finite number of parameters and has a critical point at 0 of multiplicity μ for all values of the parameters. Then every basis of the local algebra that correspond to the null value of the parameters is also a basis of the local algebra for $F(x, \lambda)$, and $\lambda \in B(0, \varepsilon)$ for an $\varepsilon > 0$ (that is, it remains a basis for small values of the parameters).*

Proof. Because of the Tougeron's finite determinacy theorem 2.3.3, we know that for all the functions in the family are $\mu + 1$-determined, so we might prove that a basis of $J^{\mu+1}(\mathbb{C}^n, \mathbb{C})/I_{\nabla(F(x,0))}$ remains a basis of the local algebra of the functions given by (sufficiently) small values of the parameters. Indeed, the isomorphism

$$I_{\nabla(F(x,0))} \oplus J^{\mu+1}(\mathbb{C}^n, \mathbb{C})/I_{\nabla(F(x,0))} \simeq J^{\mu+1}(\mathbb{C}^n, \mathbb{C})$$

tells us that for small values of the parameters, both summands will still be in direct sum (because both depend continuously on the parameters) and that a basis for the local algebra will still be a transversal to $I_{\nabla(F(x,\lambda))}$ for small values of λ, and therefore will still be a basis of the local algebra. □

Corolary 4.2.11. *If $f = f_0 + f'$ is semiquasihomogeneous with quasihomogeneous part f_0 and $e_1, \ldots, e_\mu$ form a basis for the local algebra of f_0, then $e_1, \ldots, e_\mu$ form a basis for the local algebra of f.*

Proof. Consider the function

$$f_t(x) = f_0 + \frac{1}{t^d} f'(T_t(x)) = \frac{1}{t^d} f(T_t(x)), \text{ where } T_t(x) = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n).$$

The second summand depends continuously on t since the order of f' is greater than d. Using the previous lemma, a basis of the local algebra of f_0 is also a basis of the local algebra of f_t for a small t.

As $t^d f_t = f \circ T_t$, we know that $\langle \nabla f_t \rangle = \langle \nabla f \rangle$. As T_t is a diffeomorphism, we know that it sends a basis of the local algebra of f_t to a basis of the local algebra of f. Also, every monomial is sent by T_t to another one that is proportional to it, so the basis of $Q_{\nabla(f_t)}$ is also a basis of $Q_{\nabla(f)}$ (and it was a basis of $Q_{\nabla(f_0)}$). □

Remark 4.2.12. The number of basis monomials of the local algebra of any (semi) quasihomogeneous function f of a fixed quasi-degree δ does not depend on the choice of the basis for the local algebra. This is because that number is equal to the dimension of the space

$$A_\delta / (A_{<\delta} + A_\delta \cap I_{\nabla f}).$$

Corolary 4.2.13. *Any two semiquasihomogeneous functions of fixed degree d and weights α have the same number of basis monomials of the local algebra of a fixed degree δ.*

Proof. Because of 4.2.8, it is enough to consider only nondegenerate quasihomogeneous functions. It is easy to see that the set of nondegenerate semiquasihomogeneous functions of a fixed degree d and weights α is path connected. Along a path connecting two points of this set, the number of basis monomials of the local algebra of a given degree δ is locally constant because of 4.2.10. Then, it is constant along the curve. □

Definition 4.2.14. A map $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with components $F_1, \ldots, F_n : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be quasihomogeneous of degree $d = (d_1, \ldots, d_n) \in (\mathbb{Z}_{\geq 0})^n$ and type $\alpha \in (\mathbb{Q}_{\geq 0})^n$ if the function-germs F_i are quasihomogeneous of degree d_i and weights α for every $1 \leq i \leq n$.

The map F is said to be semiquasihomogeneous if $F = F_0 + F'$ where F_0 is a nondegenerate quasihomogeneous map-germ and F'_i (the i-th component of F') has order greater than the degree of the corresponding component $(F_0)_i$ of F_0 for every $1 \leq i \leq n$.

Example 4.2.15. If f is a quasihomogeneous function of degree d and weights $(\alpha_1, \ldots, \alpha_n)$, then $\nabla(f)$ is a quasihomogeneous map of weights $(\alpha_1, \ldots, \alpha_n)$ and degree $d = (d, \ldots, d) - \alpha$ as seen in 4.2.7.

Proposition 4.2.16. *Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a quasihomomgeneous map of multiplicity μ with integer-valued weights α and degrees $d = (d_1, \ldots, d_n)$ (for any quasihomogeneous map, we can multiply all these rational numbers by a common denominator and always obtain such weights). Consider the map $T : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $T(y_1, \ldots, y_n) = (y_1^{\alpha_1}, \ldots, y_n^{\alpha_n})$. Then $F \circ T$ has for its components homogeneous functions (in the ordinary sense) of degrees $d_1, \ldots, d_n$ and its multiplicity is $\mu \prod_{s=1}^n \alpha_s$. In fact, if $e_1, \ldots, e_\mu$ is a monomial basis of Q_F, then a monomial basis for $Q_{F\circ T}$ is $\left\{e'_{i,a} = T^*(e_i)y^a : 1 \leq i \leq \mu, a \in (\mathbb{Z}_{\geq 0})^n, 0 \leq a_s \leq \alpha_s\right\}$.*

Proof. In the i-th component of F, a monomial x^k of quasi-degree d_i and weights α determines a monomial $x^{(k_1\alpha_1, \ldots, k_n\alpha_n)}$ of degree $\sum_{s=1}^n k_s\alpha_s = \langle k, \alpha\rangle = d_i$. For the formula of the multiplicity, consider the set $\{y : F \circ T(y) = \varepsilon\}$ for ε a small regular value. Since $\{z : F(z) = \varepsilon\}$ has μ non-zero solutions $c_1, \ldots, c_{\mu(F)}$ (by 2.2.9) and for each of them, the system $T(w) = c_i$ has $\prod_{s=1}^n \alpha_s$ solutions, we conclude that the set $\{y : F \circ T(y) = \varepsilon\}$ has $\mu \prod_{s=1}^n \alpha_s$ elements and thus $\mu(F \circ T) = \mu \prod_{s=1}^n \alpha_s$. Finally, we will see that the monomials $e'_{i,a}$ generate $Q_{F\circ T}$. Take a map $g : (\mathbb{C}^n, 0) \to \mathbb{C}$. We can write it as $g(y) = \sum_{a:0\leq a_s\leq\alpha_s} y^a T^*(g_a)$ with $g_a : (\mathbb{C}^n, 0) \to \mathbb{C}$ holomorphic. Since each g_a can be reduced to the form $\sum_{i=1}^\mu c_{i,a}e_i + \sum_{s=1}^n F_s h_{s,a}$, we get that

$$g(y) = \sum_{a:0\leq a_s\leq\alpha_s} y^a T^*(g_a) = \sum_{a:0\leq a_s\leq\alpha_s} \sum_{i=1}^{\mu} c_{i,a} y^a T^*(e_i) + \sum_{a:0\leq a_s\leq\alpha_s} \sum_{s=1}^{n} T^*(F_s) y^a h_{s,a}.$$

So, the $e'_{i,a}$ generate $Q(F \circ T)$ and as they are exactly $\mu(F) \prod_{s=1}^n \alpha_s$ monomials, they form a basis. □

Definition 4.2.17. The *Poincaré polynomial* of a semiquasihomogeneous map F of given integral and coprime weights α is the polynomial $P_F(t) = \sum \mu_i t^i$ such that every μ_i is the number of basis monomials of Q_F of quasidegree i. We can always assume that the weights are integer by multiplying the weights by a common denominator. Also, we can assume that the weights are coprime by dividing them by a common factor.

Remark 4.2.18. $\mu(F) = P_F(1)$.

To understand the behaviour of the basis of Q_F, we have a formula for its Poincaré polynomial.

Theorem 4.2.19. *The Poincaré polynomial of a semi-quasihomogeneous map F of degree d and integral and coprime weights α is given by the formula*

$$P_F(t) = \prod_{s=1}^{n} \frac{t^{d_s} - 1}{t^{\alpha_s} - 1}.$$

Proof. Because of 4.2.8, it is enough to consider the case of a nondegenerate quasihomogeneous map F. By making a change of variables T like in the last proposition 4.2.16, and using the form of the basis monomials of $F \circ T$, we deduce that

$$P_{F\circ T}(t) = P_F(t)P_T(t)$$

where both $F \circ T$ and T are considered homogeneous in the usual degree (that is, with weights $(1, \dots, 1)$). The polynomials P_T and $P_{F\circ T}$ are easier to compute.

Since the the Poincaré polynomial for the map $T_1 : (\mathbb{C}, 0) \to \mathbb{C}$ such that $T_1(x) = x^n$ is $\frac{t^n-1}{t-1}$ and a basis of monomials of Q_T is formed by the polynomials $\left\{x^k : 0 \leq k_i < \alpha_i\right\}$, we get that $P_T(t) = \prod_{i=1}^{n} \frac{t^{\alpha_i}-1}{t-1}$. Also, $F \circ T$ is a nondegenerate map, whose components $(F \circ T)_i$ are homogeneous functions of degrees D_i. Consequently, applying 4.2.13, it has the same Poincaré polynomial as any other map with the same degrees. Since we already computed that polynomial for the map $T'(x_1, \dots, x_n) = (x_1^{d_1}, \dots, x_n^{d_n})$, we obtain

$$P_{F\circ T}(t) = P_{T'}(t) = \prod_{i=1}^{n} \frac{t^{d_i} - 1}{t - 1}.$$

Finally, $P_F(t) = \frac{P_{F\circ T}}{P_T} = \prod_{i=1}^{n} \frac{t^{\alpha_i}-1}{d_i-1}$. ☐

Corolary 4.2.20. *Under the same hypothesis of the previous theorem, and naming μ to the multiplicity of F and $d_{\max}$ the higher quasidegree of all the basis monomial of Q_F,*

1. $\mu = \prod_{i=1}^{n} \frac{d_i}{\alpha_i}$
2. $d_{\max} = \sum_{i=1}^{n}(d_i - \alpha_i)$ *and there is only one basis monomial of Q_F of quasidegree $d_{\max}$.*

Proof. Because of the theorem, $P_F(t) = \prod_{s=1}^{n} \frac{t^{d_s}-1}{t^{\alpha_s}-1} = \prod_{s=1}^{n} \frac{t^{d_s}-1}{t-1}\frac{t-1}{t^{\alpha_s}-1}$. The first equality shows that P_F is monic of degree $\sum_{i=1}^{n}(d_i - \alpha_i)$. Since $\frac{t^n-1}{t-1} = 1 + t + \dots + t^{n-1}$, when we evaluate $t = 1$ in $\prod_{s=1}^{n} \frac{t^{d_s}-1}{t-1}\frac{t-1}{t^{\alpha_s}-1}$ we get $\mu = P_F(1) = \prod_{s=1}^{n}(d_s - \alpha_s)$.

☐

Definition 4.2.21. A formal vector field $v = \sum v_i \frac{\partial}{\partial x_i}$ has *order* d of weights α if differentiating in the direction of the field v raises the order of a function in at least d, that is $v(A_\lambda) \subseteq A_{\lambda+d}$. We denote $\mathfrak{g}_d$ to the set of vector fields of order d. This induces a filtration in the module of vector fields that is compatible with the filtration of the algebra:

$$f \in A_d, v \in \mathfrak{g}_{d'} \Rightarrow fv \in \mathfrak{g}_{d+d'}, v(f) \in A_{d+d'}.$$

Remark 4.2.22. Let $f \in A_\lambda, v_1 \in \mathfrak{g}_d, v_2 \in \mathfrak{g}_{d'}$, then $L_{v_1}L_{v_2}(f) - L_{v_2}L_{v_1}(f) \in A_{\lambda+d+d'}$. This says that the Poisson bracket determines a Lie algebra structure on every $\mathfrak{g}_d$ and that every $\mathfrak{g}_d$ is an ideal in the algebra $\mathfrak{g}_0$.

Definition 4.2.23. Let $x = (x_1, \dots, x_n)$ and $k \in (\mathbb{Z}_{\geq 0})^n$. A *vector-monomial* is a vector field of the form $x^k \frac{\partial}{\partial x_i}$. Its degree is $\langle k - 1_i, \alpha \rangle \in \mathbb{Q}$ where 1_i is the i-th vector of the canonical basis of $\mathbb{Z}^n$ (note that its degree can be negative). A vector field is quasihomogeneous of degree d if all the monomials with non-zero coefficient have degree d.

Proposition 4.2.24. *A field $v = \sum v_i \frac{\partial}{\partial x_i}$ of weights α has order d if and only if each of its components v_i is a function-germ of order $d + \alpha_i$.*

Proof. If $v \in \mathfrak{g}_d$, then $v_i = L_v(x_i) \in A_{d+\alpha_i}$, because $x_i \in A_{\alpha_i}$.

Now, let $v_i = \sum_{k \in (\mathbb{Z}_{\geq 0})^n} v_{i,k} x^k$ of order $d + \alpha_i$. Then, for every monomial $f = x^l$ we have

$$L_v(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i} = \sum_{k \in (\mathbb{Z}_{\geq 0})^n} \sum_{i=1}^n l_i v_{i,k} x^{l+k-1_i}.$$

Now, $\langle l + k - 1_i, \alpha \rangle \geq \langle l, \alpha \rangle + d$ because $\langle k, \alpha \rangle \geq d + \alpha_i$ for every k exponent of a monomial with non-zero coefficient in v_i. This says that $L_v A_\lambda \subseteq A_{\lambda+d}$ for every $\lambda \geq 0$. □

Consider the local algebra of a semiquasihomogeneous function f of degree d and fix a system of monomials forming a basis for its local algebra.

Definition 4.2.25. A monomial is said to be *upper* or *lying above the diagonal* (respectively, *lower* or *diagonal*) if it has degree greater than d (respectively, less tan d or equal to d) for the given type of quasihomogeneity. As we know from 4.2.8, the number of upper, diagonal or lower basis monomials does not depend on the choice of the basis.

Lemma 4.2.26. *Let F be a power series of order d and v a vector field of positive order δ, for fixed weights of quasihomogeneity α. Then the Taylor formula*

$$F(x + v(x)) = F(x) + \frac{\partial F}{\partial x} v + R$$

holds, where R is a remainder term of order greater than $d + \delta$.

Proof. It is enough to show it for $F = x^k$ and $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ (because of the linearity of the expression). We can express $v_i = \sum_{k=1}^\infty \sum_{l \in (\mathbb{N}_0)^n : |l| = k} v_{i,l} x^l$ for every $1 \leq i \leq n$. By writting the Taylor expansion of F, each of the monomial terms that arise from the summand $\frac{\partial^{|m|} f}{\partial m} v_1^{m_1} \cdots v_n^{m_n}, m = (m_1, \dots, m_n) \in (\mathbb{N}_0)^n$ have exponents of the form $k - m + \sum_{j=1}^{m_i} \sum_{i=1}^n l_{i,j}$ where each $l_{i,j}$ is the exponent of one of the monomials in the Taylor series of v_i. Therefore, each of these exponents can be written as

$$k - m + \sum_{j=1}^{m_i} \sum_{i=1}^n l_{i,j} = k + \sum_{j=1}^{m_i} \sum_{i=1}^n (l_{i,j} - 1_i)$$

where $1_i \in (\mathbb{N}_0)^n$ has a 1 in the i-th coordinate and zeros in the rest of them. Since our hypothesis say that

$$\langle k, \alpha \rangle \geq d \text{ and } \langle l_{i,j} - 1_i, \alpha \rangle \geq \delta > 0$$

we conclude that the degree of each of these monomials is greater than $d + |m|\delta$. So, for every term with $|m| > 1$, we know it has order strictly greater than $d + 2\delta$, which is what we had to prove.

□

Theorem 4.2.27 (Normal forms for semiquasihomogeneous functions). *Let f_0 be a quasihomogeneous function and $e_1, \dots, e_s$ a system of all upper basis monomials of a fixed basis of its local algebra. Then, every semiquasihomogeneous function $f = f_0 + f_1$ with quasihomogeneous part f_0 is equivalent to a function of the form $f_0 + \sum_{i=1}^{s} c_i e_i$.*

Proof. The main idea of the proof is to cancel the high-order terms using the quasihomogeneous part f_0. Denote by g the sum of the terms of degree $d' > d$ in f_1. Let $e_1, \dots, e_r$ be all the monomials of degree d' on the considered basis of the local algebra. Then, we can write

$$g = \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i} v_i(x) + \sum_{j=1}^{r} c_i e_i.$$

Since $g, e_1, \dots, e_r$ is quasihomogeneous of degree d', we can choose $v = \sum v_i \frac{\partial}{\partial x_i}$ to be quasihomogeneous of degree $d' - d > 0$ (when we write v as a sum of its quasihomogeneous part, it is clear that if we replace v by its quasihomogeneous part of degree $d' - d$, the same formula holds).

Now, consider the change of variables $x = y - v(y)$. It is indeed a biholomorphism: we compute $\frac{\partial x_i}{\partial y_j}(y) = \delta^i_j - \frac{\partial v_i}{\partial y_j}(y)$. Since the function $\frac{\partial v_i}{\partial y_j}(y)$ is quasihomogeneous of degree $\delta + \alpha_i - \alpha_j$, we know it will vanish in 0 if $\alpha_i \geq \alpha_j$. So, if we permute the coordinates such that the weights are in decreasing order, the jacobian matrix at 0 will be triangular with 1s on the diagonal.

Now, using 4.2.26 on f, we get

$$f(y - v(y)) = f_0(y) + f_1(y) - \sum_{i=1}^{n} \frac{\partial f_0}{\partial x_i}(y) v_i(y) - \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(y) v_i(y) + R$$

$$f(y - v(y)) = f_0(y) + \left[f_1(y) + \sum_{i=1}^{r} c_i e_i - g(y) \right] - \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(y) v_i(y) + R.$$

Here, both R and the last sum have orders greater than d'. Thus, we were able to make our function equivalent to one with exactly the same terms of degree less than d' and changed the terms of degree d' to $\sum_{i=1}^{r} c_i e_i$ (by altering only the terms of higher order).

Using this procedure step by step in increasing order of degrees, we can obtain the required normal form modulo terms of arbitrarily high degree (or obtain it through a formal biholomorphism). To finish the proof, we use the finiteness of the multiplicity of f_0 (that has not been used yet). Indeed, by Tougeron's theorem 2.3.3, we know that any function of multiplicity μ is equivalent to its $\mu + 1$ Taylor polynomial. Thus, we apply our procedure finitely many times until all its monomials of degree less or equal than $\mu + 1$ have degree less than d', and then we use one last biholomorphism to throw the terms of (usual) degree less or equal than $\mu + 1$.

□

Example 4.2.28. Let k be an integer greater or equal than 3. Consider a semiquasi-homogeneous function f of quasihomogeneous part $f_0(x, y) = x^2y + y^k$ (of degree 1 and weights $\alpha_1 = \frac{k-1}{2k}, \alpha_2 = \frac{2}{2k}$). Because of 4.2.8, a basis of monomial of its local algebra $Q_{\nabla(f)} = \mathscr{O}_n/\langle 2xy, x^2 + ky^{k-1}\rangle$ is formed by the monomials $\{1, x, y, y^2, y^3, \ldots, y^{k-2}, y^{k-1}\}$. Thus, it has multiplicity $\mu(f) = k + 1$ and none of its basis monomials lie above the diagonal. Hence, it is equivalent to its quasihomogeneous part because of 4.2.27.

Chapter 5

Classification of singularities

Now that we have all the tools required, we proceed to make some computations to obtain all the normal forms of simple singularities. We will also give some examples of normal forms for unimodal singularities. This classification allows to generalize the description of the elementary catastrophes given by Thom in the 1960s. The simple singularities have an ADE classification, which relates the classification of simple singularities with other constructions, such as resolution of singularities of complex surfaces or Kleinian singularities. The results and the order of the exposition are taken from [2], but using in many cases, different techniques for the reduction to normal forms.

Proposition 5.0.1. *If a germ has a singularity of finite multiplicity and corank* 1*, then it is stably equivalent to a function of the form* x^n*.*

Proof. Using the Splitting lemma 1.2.5, we know that f is stably equivalent to a function in one variable $\tilde{f}(x) = \sum_{i=1}^{\infty} a_i x^i$ (we assume that $\tilde{f}(0) = 0$). If $n = \min\{i \in \mathbb{N} : a_i \neq 0\}$ (if every coefficient is equal to 0, then f does not have finite multiplicity), we know that $\tilde{f}(x) = x^n(\sum_{i=0}^{\infty} a_{n+i} x^i)$. Thus, by chosing a branch of the n-th root h well defined in a neighborhood of a_n and making the change $y = xg(\sum_{i=0}^{\infty} a_{n+i} x^i)$, we get that f is stably equivalent to the map y^n (indeed, $g(\sum_{i=0}^{\infty} a_{n+i} x^i)$ is a unit).

□

Definition 5.0.2. If a singularity is stably equivalent to the function x^{k+1} for $k \in \mathbb{N}$, we say that it has type $\mathbf{A}_k$.

This classifies all the functions with corank 1. In the case of simple singularities, they have at most corank 2, as shown in the following result.

Proposition 5.0.3. *The corank of a simple singularity is less or equal than* 2*.*

Proof. If the corank of the function-germ f is equal to k, then it is stably equivalent to a function $\varphi : (\mathbb{C}^k, 0) \to \mathbb{C}$ and $\varphi \in \mathfrak{m}^3$. The action of the groups of biholomorphisms over the 3-jets of functions in $\mathfrak{m}^3$, induces an action of $GL_k(\mathbb{C})$, in the form of linear sustitution. This is because if $g \in \mathfrak{m}^3$ and $h \in \mathfrak{m}$ is a biholomorphism, then $j^3(g \circ h) = g \circ j^1(h)$. Thus, if the cubical forms of two functions-germs lie in different orbits of the action of $GL_k(\mathbb{C})$, then they cannot be equivalent. Since the group $GL_k(\mathbb{C})$ has dimension k^2, and the space of cubical forms has dimension $\binom{k+2}{3}$, and for $k \geq 3$, we get $\binom{k+2}{3} > k^2$, it is impossible that finite orbits cover a small neighborhood of $j^3(\varphi)$.

□

Remark 5.0.4. From now on, since we know that the corank of the simples singularities is less or equal to 2, we will always take a member φ of the equivalence class with $(\mathbb{C}^2, 0)$ as a source space that is contained in $\mathfrak{m}^3$. The taylor expansion of φ begins from cubic terms. Say that its cubic part is the homogeneous polynomial $ax^3 + bx^2y + cxy^2 + dy^3$. Thinking of it as function in $\mathbb{C}P^1$, we have an associated cubic polynomial $a + bt + ct^2 + dt^3 = (t - \alpha_1)(t - \alpha_2)(t - \alpha_3)$ Thus, after making a linear change of coordinates, we can make every polynomial equivalent to one of the following:

1. $P(x, y) = x^2y + y^3$, which corresponds to three different roots ($t = i, -i, 0$) in the associated polynomial;
2. $P(x, y) = x^2y$, which corresponds to a simple root ($t = 0$) and a double root ($t = \infty$ in $\mathbb{C}P^1$);
3. $P(x, y) = x^3$ which corresponds to a triple zero;
4. $P \equiv 0$.

Thus, the initial form of any corank 2 simple singularity can be made equivalent under a linear change of coordinates to one of the list. Actually, if the singularity is simple, the last option is not available, as we will prove now.

Proposition 5.0.5. *If a function-germ $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ of corank 2 has a simple singularity, then its cubical form is not equal to* 0*, that is, $f \notin \mathfrak{m}^4$.*

Proof. If $f \in \mathfrak{m}^4$, then its 4-jet $j^4(f)$ is a homogeneous polynomial or 0, and in the first case it vanishes along 4 lines intersecting in the origin. Having in mind that the action of biholomorphisms induces an action of $GL_2(\mathbb{C})$ in the 4-jets, that differentiates the orbits (as in 5.0.3), the cross ratios of these lines is an invariant of the action of the diffeomorphisms on the 4-jets. Therefore, there is a 1-parameter family of function-germs that does not have two equivalent germs. This means that the singularity cannot be simple.

□

If the cubic part of (the taylor expansion of) f is $x^2y + y^3$, we can reduce our function to its cubic part via a biholomorphism.

Theorem 5.0.6. *Let $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ be a (representative of the class of a) singularity of corank* 2*, such that its cubic part $x^2y + y^3$. Then f is equivalent to its cubic part.*

Proof. First, we make a change of coordinates $(x_1, y_1) = (x + \phi(x, y), y + \psi(x, y))$, where $\phi, \psi \in \mathfrak{m}^2$. Its differential matrix is $Id + D(\phi, \psi)(x, y)$ that is invertible because the derivatives of ϕ and ψ all belong to $\mathfrak{m}$ (using 1.3.1). So, after making that change of coordinates, we have

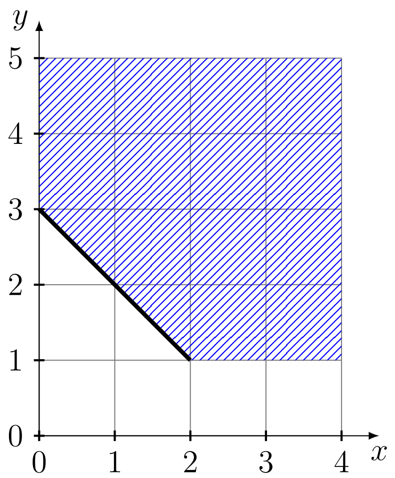
$$x_1^2y_1 + y_1^3 = (x + \phi)^2(y + \psi) + (y + \psi)^3 = x^2y + y^3 + 2xy\phi + (x^2 + 3y^2)\psi + O(5)$$

where $O(5)$ are the terms of the expansion that belong to $\mathfrak{m}^5$. So basicly, after this change of coordinates, the cubic part remains the same, and we can add any polynomial of degree 4 that belongs to $I_{\nabla(x^2y+y^3)} = \langle 2xy, x^2+3y^2\rangle = \langle xy, x^3, y^3\rangle$. Since any polynomial of degree 4 is contained in $I_{\nabla(x^2y+y^3)}$, we can cancel the terms of degree 4 in the taylor expansion of f by means of this change of coordinates. By doing the same with $\phi, \psi \in \mathfrak{m}^3$, we can cancel the terms of degree 5, keeping the 4-jet unchanged. So, our map's 5-jet will be (after both changes) equivalent to $x^2y + y^3$. Since $\mu(f) = 4$, by Tougeron's theorem 2.3.3, that says that f is equivalent to $x^2y + y^3$.

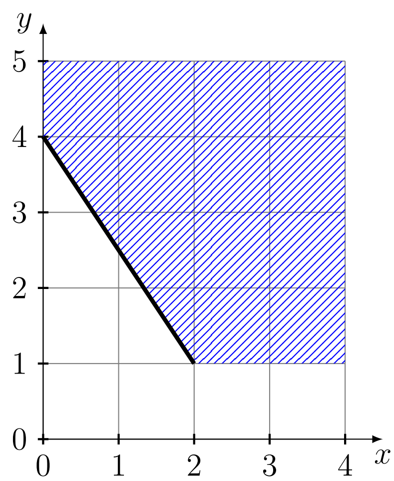
□

Remark 5.0.7. We would like to use the same argument in the general case. However, not always f is equivalent to its principal part. Indeed, if f has an isolated singularity and its cubic part is x^2y, it will never be equivalent to its cubic part (because it does not have an isolated singularity in 0). So, we can "relax" the notion of principal part of a function and try to find "another candidate to principal part" (under this new relaxed notion), to make the function f equivalent to it; this is when the concept of quasihomogeneous functions with different weights plays its role. Indeed, the fact that in the last theorem we could cancel the high degree monomials was because there were no basis monomials of $Q_{\nabla(f)}$ over the diagonal as in 4.2.27.

In general (as seen in the case of the previous theorem) it is a good idea to consider as the principal part of a function, the monomials lying over one of the segments that form the newton diagram of our function-germ (that is, considering the weights such the diagonal is parallel to that segment). Indeed, this is what we did in the last theorem (the only segment forming the newton diagram of f was indeed $\{(i, j) \in (\mathbb{N}_0)^2 : i + j = 3\}$).



(a) Newton diagram and polyhedron of $x^2y + y^3$. Here, the diagonal using the usual degree of polynomials is an edge of the Newton diagram.



(b) Newton diagram and polyhedron of x^2y+y^4. Here, the diagonal using the usual degree of polynomials is not an edge of the Newton diagram, so we change the type of quasihomogeneity to make the diagonal an edge.

Figure 5.1

Example 5.0.8. Let us see the case of the functions having the segment joining $(0, k)$ and $(2, 1)$ as an edge of the Newton diagram. In 5.1b, it is drawn for $k = 4$.

We know that this function is indeed a semi-quasihomogeneous function with principal part $f_0(x, y) = x^2y + y^k$. As we saw before in 4.2.28, $\{1, x, y, y^2, y^3, \dots, y^{k-2}, y^{k-1}\}$ is a basis of the local algebra, so its an isolated singularity. In this case, the cancellation of higher order terms is done by virtue of the Theorem 4.2.27: since neither of the monomials of the basis of the local algebra lie above the diagonal, we can make any function with that Newton diagram equivalent to f_0.

Remark 5.0.9. However, in general it is not true that any function can be chosen to be semi-quasihomogeneous. Indeed, consider the case of $f(x, y) = x^6 + x^2y^2 + y^6$. The possible principal parts are considered to be $x^6 + x^2y^2$, x^2y^2 or $x^2y^2 + y^6$, depending on the type of quasihomogeneity used. In neither of those cases, the singularities is isolated while $Q_{\nabla(f)} = \mathbb{C}[x, y]/\langle 6x^5 + 2xy^2, 6y^5 + 2yx^2\rangle$ has finite dimension. This is because every monomial $\overline{ax^iy^j} \in Q_{\nabla(f)}$ has a monomial $\overline{bx^\alpha y^\beta}$ with bidegree $\alpha+\beta \le 10$ as a representative (otherwise, α or β will be greater than five and we can replace it with an equivalent polynomial of smaller bidegree).

Definition 5.0.10. We say that a singularity is of type $\mathbf{D}_{k+1}$ if it is stably equivalent to the function $x^2y + y^k$.

Continuing the classification of simple singularities, assume that the 3-jet of a simple function-germ f is equivalent to x^2y. In this case, we would like to take one of the edges of its Newton diagram and use our theorem of reduction to normal forms of semiquasihomogeneous functions 4.2.27. As x^2 cannot divide f (because it is an isolated singularity), we know that in its Taylor series there are monomials having non-zero coefficient with degree in x less than 2. In the Newton diagram 5.2a, it means that in the blue area, there must be an integral point that belongs to its Newton diagram. Since we want to "find" all the simple singularities with 3-jet equivalent to x^2y, the idea will be to "start" with a line $\{i + j = 3\}$ and rotate it clockwise around the point $(2, 1)$ until we "crush" a point (i_0, j_0) that corresponds to a monomial with non-zero coefficient in the Taylor series of f (this happens since the blue area of 5.2a contains one of these). This method is called *Newton's rotating ruler method*, depicted in 5.2. After rotating the ruler, we find that there are two possibilities:

1. The ruler first strikes only a point $(0, k)$. In that case, the Newton diagram of f will have an edge joining only $(0, k)$ and $(2, 1)$. This says that the function f is semiquasihomogeneous with quasihomogeneous part $Ax^2y + By^k$ with $A, B \neq 0$ (obviously, this is equivalent to $x^2y + y^k$). 4.2.28 says that it is equivalent to its quasihomogeneous part $x^2y + y^k$.

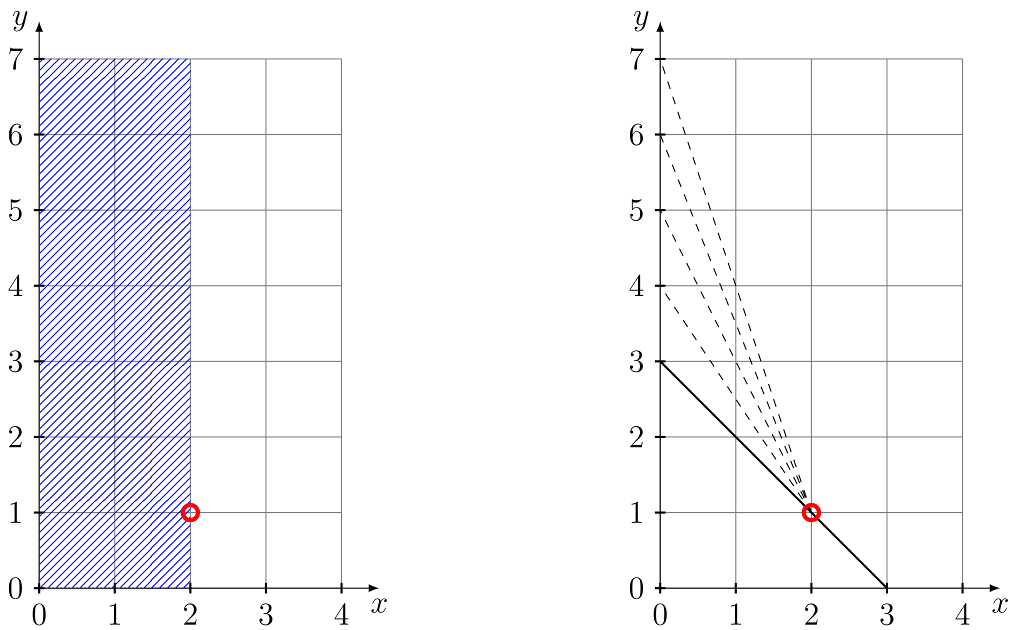
2. The ruler first strikes two points corresponding to the polynomials xy^{k+1} and y^{2k+1}. In this case, the polynomial determined by the points touching the ruler is

$$Ax^2y + Bxy^{k+1} + Cy^{2k+1}, A \neq 0.$$

In this case, the change $x' = x - \frac{B}{2A}y^k, y' = y$ gives

$$A(x^2y - \frac{B}{A}xy^{k+1} + \frac{B^2}{4A^2}y^{2k+1}) + B(xy^{k+1} - \frac{B}{2A}y^{2k+1}) + Cy^{2k+1}$$
$$= Ax^2y + (C - \frac{B^2}{4A})y^{2k+1}.$$

If $B^2 \neq 4AC$, then we can make this map equivalent to a semiquasihomogeneous one of quasihomogeneous part of type D_k, reducing it to the first case. Otherwise, after making this change, we obtain an equivalent map that has no monomials below the ruler, so we keep rotating the ruler until we strike another monomials. We repeat this operation, and if the function has finite multiplicity, it should end in finite steps (that means, we should get it reduced to the first case).



(a) The blue zone must have a monomial or the singularity will not be isolated.

(b) We rotate the ruler, starting from the black segment.

Figure 5.2: Newton's rotating ruler method.

In the same direction, we will find normal forms for the functions with cubical form x^3.

Lemma 5.0.11. *A simple germ of a function of 2 variables of corank* 2 *with cubical form* x^3 *is equivalent to one of the* 3 *normal forms:*

- $E_6 : x^3 + y^4$
- $E_7 : x^3 + xy^3$
- $E_8 : x^3 + y^5$.

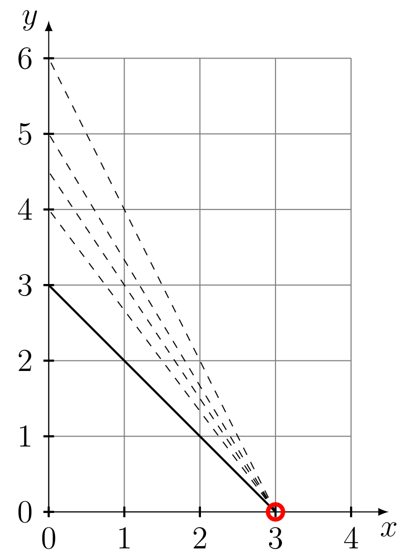


Figure 5.3: Once again, we rotate the ruler, starting from the black segment.

Proof. In the same spirit, we begin by using our method of rotating the ruler around x^3. If we strike firstly one of the monomials y^4, xy^3 or y^5, we know that the function is semiquasihomogeneous and does not have upper monomials in its local basis (the basis of each local algebra is written in 5.0.12). Therefore, we can make our function equivalent to one of our normal forms using the Theorem 4.2.27.

Otherwise, if the coefficient of xy^4 is not zero, the 5-jet of the function is equivalent to $x^3 + xy^4$. Indeed, if it is of the form $x^3 + axy^4 + 3x^2\varphi$ with $\varphi \in \mathfrak{m}^2$, we make the change $x_1 = x-\varphi$ to cancel the term with φ and then multiply y by a scalar. Now, our new function should be semiquasihomogeneous with quasihomogeneous part $f_0(x, y) = x^3 + xy^4 + \lambda y^6$ for $\lambda \in \mathbb{C}$. Indeed, the local algebra of the quasihomogeneous part is

$$Q_{f_0} = \mathscr{O}_2/\langle 3x^2 + y^4, 4xy^3 + 6\lambda y^5\rangle.$$

It is obviously finitely generated, and its basis are $\langle 1, x, xy, xy^2, y, y^2, y^3, y^4, y^5.y^6\rangle$ if $\lambda = 0$ and $\langle 1, x, x^2, xy, xy^2, y, y^2, y^3\rangle$ if $\lambda \neq 0$. In any case, there are no upper monomials in neither basis of the respective local algebras, and thus can be made equivalent to its quasihomogeneous part. Now, let $f_\lambda(x, y) = x^3 + xy^4 + \lambda y^6$ be a 1-parameter family. We will show that the orbit of the germ at 0 of f_λ varies continuously with λ. This will show that this germ cannot be simple (if the coefficient of xy^4 vanishes, we can make it a two-parameter family and show that there is a curve in the space of parameters where the orbit varies continuously).

Note that the zeros of f_λ form 3 parabolas of the form $x = k_i y^2$, where $k_i, i = 1, 2, 3$ are the three roots of the equation $k^3 + k + \lambda = 0$. We will show that the ratio $\frac{k_3-k_1}{k_2-k_1}$ (defined after ordering the roots by its imaginary part) is an invariant via the action of biholomorphisms. It is enough to show that a biholomorphism that carries the ordered triple $x = 0, x = y^2, x = my^2$ into other triple of the same form, has different m, that is, if the triples $x = 0, x = y^2, x = my^2$ and $x = 0, x = y^2, x = m'y^2$ are carried from one to the other by a biholomorphism, then $m = m'$ (we are making $k_1 = 0, k_2 = 1$). Indeed, we can transform two of the three parabolas in the pair $x = 0, x = y^2$ by making the change $x' = x - k_1y^2, y' = \sqrt{k_2 - k_1}y$.

Since a biholomorphism h that carries one triple to the other leaves the y axis fixed, it must have the form

$$h_1(x,y) = x(\alpha + u(x,y)), h_2(x,y) = \beta x + \gamma y + v(x,y) \text{ where } u \in \mathfrak{m}, v \in \mathfrak{m}^2.$$

Since the image of $x = y^2$ (in its 2-jet) must be $h_1 = h_2^2$, we must have that $\gamma^2 = \alpha$. And thus, the image of the parabola $x = my^2$ is $x = my^2$ again since

$$h_1(x,y) = x(\alpha + u(x,y)) = my^2(\gamma^2 + u(x,y)) \text{ in the curve } x = my^2$$
$$m\left(h_2(x,y)\right)^2 = m\left(\beta y^2 + \gamma y + v(x,y)\right)^2 = \gamma^2 y^2 + O(y^3) \text{ in the curve } x = my^2.$$

This says that the family varies continuously, since the ratio $\frac{k_3-k_1}{k_2-k_1}$ (where k_1, k_2, k_3 are the three roots of $k^3 + k + \lambda$) varies continuously with λ and is equal to -1 if and only if $k_3 + k_2 = 2k_1$ (which means that $k_1 = 0$, since $k_1 + k_2 + k_3 = 0$). Thus, we have reduced the normal forms simple singularities with 3-jet equivalent to x^3, to E_6, E_7 and E_8.

□

Remark 5.0.12.

- The miniversal deformation of $x^k, k \in \mathbb{N}$ is $x^k + \sum_{i=0}^{k-2} \lambda_i x^i$. Thus, the singularity is simple since every deformation of it belongs to one of the finite orbits $x, x^2, \ldots, x^{k-1}$ for $1 \le j \le k-1$ (in the smaller space $\mathfrak{m}$). See 5.0.1.

- The miniversal deformation of $x^2y + y^k, k \in \mathbb{N}_{\ge 3}$ is $x^2y + y^k + \lambda_k x + \sum_{i=0}^{k-1} \lambda_i y^i$, as shown in 4.2.28. The corank of a function-germ defined by a fixed set of parameters, has corank 1 or 2, depending on the coefficient λ_2 (assuming that $\lambda_k = 0$, or otherwise 0 is not a critical point). In that case, it is clear that the local algebra cannot have dimension higher than k. In any other case, since the function has the monomial x^2y in its development, it is equivalent to one of the D_k (as we shown in our Newton's rotating the line method). It is also clear in this case that the dimension of the local algebra is less or equal than k. After cracking all these cases, we get that this singularity is simple, since only finitely many orbits can be intersected after a small deformation.

- In the case of $x^3 + y^4$, its local algebra is generated by $1, x, y, y^2, xy, xy^2$, so it has multiplicity 6 and cannot be perturbed to have higher multiplicity. Depending on the corrank of it, we can make it equivalent to A_k, D_k with $k \le 5$ (by rotating the ruler as we did before).

- The cases of E_7 and E_8 are similar to E_6. The versal deformations can be found in 5.0.12. Making analogous computations to the ones showed in the case of $D_k, k \in \mathbb{N}$ and E_6, we can show that they are simple. Indeed, any sufficiently small deformation of them can be made equivalent to $A_k, D_k, k \le 7$ or E_6 in the case of E_7 and to $A_k, D_k, k \le 8$, E_6 or E_7 in the case of E_8.

Thus, we have proved that all the simple singularities are those listed below.

Theorem 5.0.13. *If f is a simple singularity, then it is stably equivalent to one of the following singularities:*

- $\mathbf{A}_k : x^{k-1}$

- $\mathbf{D}_k : x^2y + y^{k-1}$
- $\mathbf{E}_6 : x^3 + y^3$
- $\mathbf{E}_7 : x^3 + xy^3$
- $\mathbf{E}_8 : x^3 + y^4$.

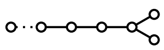
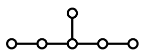
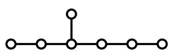
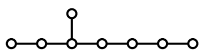
Remark 5.0.14. A particular case of our theorem is that in families of 4 parameters or less, one meets only with the singularities $A_n, n \leq 5$, D_4 and D_5. This assertion is usually known as “Thom’s theorem” or also “Thom’s rule of the seven elementary catastrophes”, and is one of the fundamental results in Catastrophe Theory. See for example [5] Chapter 5, [4] Chapter 15, [7] Chapters 7 and 9.

Remark 5.0.15. The fact that the series of simple singularities have that name is not by chance. There is a deep conection with other objects in other areas of mathematics, having also this ADE classification. We will glimpse this relation by showing one example. Finding relations between objects that have the ADE classification has been a major topic of study in several areas (especially in representation theory).

Let

$$G_{n+1} = \left\{ \begin{pmatrix} \zeta_{n+1}^k & 0 \\ 0 & \zeta_{n+1}^{-k} \end{pmatrix} : 0 \leq k \leq n, \zeta_{n+1} \text{ (n+1)-th primitive root of 1 } \right\}$$

be the cyclic finite subgroup of SU_2 (which we identify with $\mathbb{Z}/(n+1)\mathbb{Z}$). The quotient space given by $\mathbb{C}^2/G_{n+1}$ can be identified with $\mathbb{C}\left[u,v\right]^{G_{n+1}}$, that is the polynomials in two variables fixed by the action of the group. Since the element $k \cdot u^i v^j$ is $\zeta^{k(i-j)}u^i v^j$, the monomials fixed by the action are exactly $\{u^i u^j : i - j \equiv 0 (\mathrm{mod}\ n)\}$. That is, they are generated by the elements xy, x^{n+1}, y^{n+1}. Since the kernel of the morphism $\mathbb{C}[a,b,c] \to \mathbb{C}[uv, u^{n+1}, v^{n+1}]$ is $\langle a^{n+1} - bc \rangle$, we get that it is isomorphic to $\mathbb{C}[a,b,c]/\langle a^{n+1} - bc\rangle$, and making the change $x = a, y = (b+ic), z = (b-ic)$, it is isomorphic to $\mathbb{C}[x,y,z]/\langle x^{n+1} + y^2 + z^2\rangle$. The factoring ideal is generated by a polynomial that is exactly the normal form A_n. Moreover, the list of finite subgroups of SU_2 is given by the dihedral group $\mathbb{D}_{2n}$ and the groups of symetries and the binary tetrahedral, octahedral and icosahedral groups (named $\mathbb{T}$, $\mathbb{O}$ and $\mathbb{I}$ respectively). Making analogous procedures with the other finite subgroups, the factoring ideals that rise are $x^2y + y^{n+1} + z^2$ for $\mathbb{D}_{2n}$, $x^3 + y^4 + z^2$ for $\mathbb{T}$, $x^3 + xy^3 + z^2$ for $\mathbb{O}$ and $x^3 + y^5 + z^2$ for $\mathbb{I}$, which correspond to all the normal forms of simple singularities. Also, since any finite subgroup $\Gamma \subseteq SU_2 \subseteq GL_2(\mathbb{C})$ is a representation, we can build its McKay Graph: in each case, it is the corresponding Dynkin diagram.

Dynkin diagram	Finite subgroup of SU_2	Relations between the generators	Normal forms of simple singularities
	$\mathbb{Z}_{n+1}$	$x^{n+1} + y^2 + z^2$	A_n
	$\mathbb{D}_{2n}$	$x^2y + y^{n+1} + z^2$	D_{n+2}
	$\mathbb{T}$	$x^3 + y^4 + z^2$	E_6
	$\mathbb{O}$	$x^3 + xy^3 + z^2$	E_7
	$\mathbb{I}$	$x^3 + y^5 + z^2$	E_8

The reader that wishes to explore this connection (and other construction with simple singularities that are related with other ADE objects) can see [8], Chapter 4 (for the construction of Dynkin diagrams of singularities); [1] Chapter 1, Section 2 and the papers [2] and [6].

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